Pergamon

Int. J. Solids Structures Vol. 35, No. 25, pp. 3363–3389, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(98)00009–2

HYPERELASTIC MODELS FOR ELASTOPLASTICITY WITH NON-LINEAR ISOTROPIC AND KINEMATIC HARDENING AT LARGE DEFORMATION

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(*Received* 29 *May* 1997; *in revised form* 2 *December* 1997)

Abstract—This work is concerned with the formulation of hyperelastic–thermodynamic-based models for associated elastoplasticity with non-linear isotropic and kinematic hardening valid for both large elastic and large plastic deformation. On this basis, one can then introduce explicitly the assumptions of (1), small incremental plastic deformation, and (2), small elastic strain, into the general model and obtain special cases whose behaviour corresponds to that of various classical hypoelastic formulations. In particular, these are obtained on the basis of two different thermodynamic formulations for kinematic hardening with respect to the intermediate configuration. The simplest of these, in which the plastic part of the free energy does not depend explicitly on the plastic deformation, leads for example to Jaumann- or Green–Naghdi-hypoelastic-type behaviour for linear kinematic hardening in simple shear. In particular, the former case is obtained in this context when the plastic spin is assumed constant and equal to zero, and the latter case when the plastic rotation is assumed constant and equal to the identity. Allowing the plastic part of the free energy to depend explicitly on the plastic deformation yields the second thermodynamic model for kinematic hardening considered in this work. Here, again in the special case of linear hardening, Oldroyd-like behaviour for the shear stress and back stress, but not for the normal stress, is obtained in simple shear. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The formulation and numerical implementation of finite elastoplasticity models involving damage and/or hardening processes in metals has traditionally been based on the assumption that the elastic deformation in such materials remains "small", justifying in turn a hypoelastic, rate-based formulation for the stress. In this regard, considerable effort has been spent over the years in attempting to find the "appropriate" objective derivative for the stress (e.g., Dienes, 1979; Atluri, 1984; Szabó and Balla, 1989). The difficulties here are at least two-fold: (1), there are many possible "objective" derivatives, and more critical, (2), most of corresponding hypoelastic formulations are not compatible with a consistent linearization of "exact" non-linear hyperelasticity (Simo and Pister, 1984; Simo and Ortiz, 1985), in part because only the stress rate, and not the corresponding elasticity tensor, varies in the process. For the purely elastic case, this issue has been discussed in detail by Perić (1992), who examined the problem from the point of view of conjugate stress and strain pairs, showing how non-linear hyperelastic formulations based on these reduce under the assumption of small elastic strain to various hypoelastic cases.

In the present work, an algorithmic formulation of associated elastoplasticity with non-linear isotropic and kinematic hardening for large elastic and large plastic deformation is carried out on the basis of a recent thermodynamic approach (Svendsen, 1998a) to the formulation of kinematic hardening models at large deformation. In this work, attention is restricted to the local algorithmic formulation and numerical integration of the material model; the formulation of the corresponding consistent tangent operator and finite element implementation will be discussed in a future paper. One form of the model has already been applied to the finite element modelling of the effect of kinematic hardening on crack

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propagation and residual stresses at the crack tip in steels which arise during cyclic loading (Arndt *et al.*, 1997).

Alternative thermodynamic formulations of kinematic hardening at large deformation can be found in, e.g., Dogui and Sidoroff (1985), Haupt (1995), Tsakmakis (1996), and Sievert (1997). Beyond the desirability of representing true elastic material behaviour, one of the advantages of such a thermodynamic-hyperelastic approach to the formulation of large deformation elastoplasticity (e.g., for metals) is that assumptions such as small incremental plastic deformation and small elastic strains are and must be introduced explicitly into the general form of the model (i.e., that holding for arbitrary elastic and plastic deformations) in the process of deriving its algorithmic form. This is of course in contrast to a hypoelastic-based formulation, in which these assumptions are "somehow" already incorporated into the material model from the start. In addition, as we show in this work, following an approach in which such assumptions must be introduced explicitly into the formulation yields additional relations between the variables of the model, in particular between the elastic and plastic rotations, as well as between the elastic rotation and plastic spin, which hold under the assumptions of small incremental plastic deformation and small elastic strain, which do not arise in a hypoelastic-based approach.

To begin, we review the basic aspects (Section 2) of a recent thermodynamic formulation of elastoplasticity with non-linear isotropic and kinematic hardening at large deformation (Svendsen, 1998a, b), in particular focusing on the thermodynamic formulation of the constitutive relations for stress and back stress. Next, the concrete form of this formulation for the case of Armstrong–Frederick kinematic hardening is introduced and discussed (Section 3). In the context of associated plasticity (Section 4), we then turn to the algorithmic formulation of the model (Section 5) via the usual exponential and backward-Euler integration. The resulting algorithmic form of the two models, which holds for large elastic and plastic deformation, is then simplified to the case of metal plasticity (Section 6) with the help of the assumptions of (1), small incremental plastic deformation, and (2), small elastic strain. After summarizing the algorithmic formulation (Section 7), and reviewing briefly some existing thermodynamic formulations for kinematic hardening (Section 8), we then turn to a comparison of these with the current formulation in the standard context of simple shear (Section 9). Finally, general aspects of these various formulations for kinematic hardening at large deformation are discussed (Section 10), and in particular their general relation to classical hypoelastic-based formulations, which are reviewed briefly in the Appendix.

Finally, a word on notation. Let bold, upper case italic letters such as $\boldsymbol{D}$, $\boldsymbol{F}$ and $\boldsymbol{T}$ represent second-order Euclidean tensors, or time-dependent fields of such tensors, and slanted sans serif characters such as C fourth-order Euclidean tensors. In particular, let $\boldsymbol{I}$ and I represent the second- and fourth-order identity tensors, respectively. The inner product $\boldsymbol{A}\cdot\boldsymbol{B}:=\mathrm{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})$ of two second-order Euclidean tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ will be used often in what follows; here, $\mathrm{tr}(\boldsymbol{A})$ represents the trace, and $\boldsymbol{A}^{\mathrm{T}}$ the transpose, of any $\boldsymbol{A}$. Besides $\mathrm{tr}(\boldsymbol{A})$, we also work with the square $\mathrm{sq}(\boldsymbol{A}):=\boldsymbol{A}\boldsymbol{A}:=\boldsymbol{A}^2$ and cube $\mathrm{cu}(\boldsymbol{A}):=\boldsymbol{A}\boldsymbol{A}\boldsymbol{A}:=\boldsymbol{A}^3$ functions on second-order Euclidean tensors $\boldsymbol{A}$. The inner product on such tensors yields in particular the magnitude $|\boldsymbol{A}|:=\sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$ of any $\boldsymbol{A}$. Finally, let $(\boldsymbol{A}):=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})$, $\mathrm{skw}(\boldsymbol{A}):=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}})$, $\mathrm{sph}(\boldsymbol{A}):=\frac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}$, and $\mathrm{dev}(\boldsymbol{A}):=\boldsymbol{A}-\mathrm{sph}(\boldsymbol{A})$, represent the symmetric, skew-symmetric, spherical, and deviatoric, parts, respectively, of any $\boldsymbol{A}$. As such, $\mathrm{sph}=\frac{1}{3}(\boldsymbol{I}\otimes\boldsymbol{I})$, and $\mathrm{dev}=\mathsf{I}-\mathrm{sph}$. For simplicity, it proves advantageous to abuse notation in this work and denote functions and their values by the same symbol. Other notations and mathematical concepts will be introduced as the need arises in what follows.

2. BASIC MODEL CONSIDERATIONS

In this work, attention is focused on the material behaviour of a single, but otherwise arbitrary, simple, non-polar material point or element of some material body. Assuming isothermal conditions for simplicity, the deformation gradient $\boldsymbol{F}$ represents the basic "external" or "observable" independent constitutive variable here. Beyond $\boldsymbol{F}$, we assume that the dependent constitutive variables depend in addition on a set ζ of deformation-like internal

variables accounting for the effects of inelastic deformation, as well as of isotropic and kinematic hardening, on the material behaviour. On this basis, the constitutive relations in the current elastoplastic context include the material frame-indifferent forms

$$\begin{aligned} \psi &= \psi(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) \\ \boldsymbol{S} &= \boldsymbol{S}(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) \end{aligned} \tag{1}$$

for the referential free energy density ψ and second Piola–Kirchhoff stress $\boldsymbol{S}$, $\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$ representing the right Cauchy–Green deformation tensor. As usual, $\boldsymbol{S}$ is related to the Kirchhoff $\boldsymbol{K} := \det(\boldsymbol{F})\boldsymbol{T}$ and Cauchy $\boldsymbol{T}$, stress tensors via $\boldsymbol{S} := \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F}^{-\mathrm{T}}$. Analogous to (1), we have the basic material frame-indifferent form

$$\dot{\zeta} = \begin{cases} \boldsymbol{0} & l(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) \leqslant 0 \\ \dot{\zeta}(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) & l(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) > 0 \end{cases} \tag{2}$$

for the evolution of ζ in a given process $(\boldsymbol{F}, \dot{\boldsymbol{F}})$ relative to a loading function l (e.g., Lubliner, 1973), *with* $l < 0$, $l = 0$ *and* $l > 0$, *respectively, corresponding to unloading, neutral, and* loading, processes, respectively. In writing (1) and (2), we have taken advantage of the fact that such variables can always be expressed in an observer-invariant fashion. As usual, the constitutive relations in (1) are rate-independent when they are positive homogeneous of degree 0 in $\dot{\boldsymbol{C}}$, and those in (2) represent such behaviour when they are positive homogeneous of degree 1 in $\dot{\boldsymbol{C}}$.

In the context of certain additional continuity and/or differentiability assumptions on the constitutive relations, as well as their rate-independence, one can show that (1) and (2) satisfy the internal dissipation rate inequality

$$\delta := \tfrac{1}{2}\boldsymbol{S} \cdot \dot{\boldsymbol{C}} - \dot{\psi} \geqslant 0 \tag{3}$$

in any process when (i), ψ is actually independent of $\dot{\boldsymbol{C}}$, and (ii), $\boldsymbol{S}$ is given by its hyperelastic form, i.e.,

$$\begin{aligned} \psi &= \psi(\boldsymbol{C}, \zeta), \\ \boldsymbol{S} &= 2\psi_{,\boldsymbol{C}}. \end{aligned} \tag{4}$$

In this case, δ reduces to

$$\begin{aligned} \delta_{\mathrm{P}} &= \delta_{\mathrm{P}}(\boldsymbol{C}, \zeta, \dot{\boldsymbol{C}}) \\ &= -\psi_{,\zeta} \cdot \dot{\zeta}, \end{aligned} \tag{5}$$

representing the so-called plastic dissipation function. Note that the system comprised of (2) and (4) is "closed" in the sense that, given concrete, physically reasonable forms for (2) and $(4)_1$, we could solve the former in a given process for ζ, and so obtain $\boldsymbol{S}$ via $(4)_2$. Since such forms are in general not available to us, however, further constitutive assumptions are necessary.

On the basis of rheological and other considerations (e.g., Dogui and Sidoroff, 1985; Maugin, 1992; Haupt, 1995; Svendsen, 1997a, b), in particular those concerning kinematic hardening, we next assume that ψ can be split into elastic ψ_{E} and inelastic ψ_{P} (i.e., plastic) parts. In addition, let ζ consist of the pair $(\boldsymbol{P}, \xi)$, where $\boldsymbol{P}$ represents† the plastic deformation or transformation, and ξ additional deformation-like internal variables having to do with isotropic and/or kinematic hardening. Assuming as usual that isotropic and kinematic

† Here, we are using Mandel's (1972, 1974) notation for this quantity. In what follows, we relate it to the usual plastic deformation "gradient" $\boldsymbol{F}_{\mathrm{P}}$.

hardening do not influence the elastic response of the material, the elastic part ψ_E does not depend on the ξ, and so takes the form

$$\psi_E = \psi_E(\boldsymbol{C}, \boldsymbol{P}) \tag{6}$$

via $(4)_1$. On the other hand, the plastic part ψ_P of ψ does not depend on the current state of deformation, i.e., on $\boldsymbol{C}$, yielding

$$\psi_P = \psi_P(\boldsymbol{P}, \xi) \tag{7}$$

again via $(4)_1$ and $\zeta = (\boldsymbol{P}, \xi)$. Special forms of this last relation have been considered by, e.g., Dogui and Sidoroff (1985), Haupt (1995) and Tsakmakis (1996) in conjunction with kinematic hardening, as will be discussed in more detail below. Together, then, (6) and (7) lead to the particular constitutive form

$$\psi(\boldsymbol{C}, \boldsymbol{P}, \xi) = \psi_E(\boldsymbol{C}, \boldsymbol{P}) + \psi_P(\boldsymbol{P}, \xi) \tag{8}$$

for ψ from $(4)_1$ with $\zeta = (\boldsymbol{P}, \xi)$.

As is the case in crystal plasticity, we next assume in the more general phenomenological context being considered here that $\boldsymbol{P}$ does not affect the form of the elastic constitutive relation $(4)_2$. As discussed in detail in Svendsen (1998a, b), this will in fact be the case when the elastic part ψ_E of ψ in (8) takes the special form†

$$\psi_E(\boldsymbol{C}, \boldsymbol{P}) = \varphi_E(\boldsymbol{P}^{-T}\boldsymbol{C}\boldsymbol{P}^{-1}), \tag{9}$$

yielding the form $\boldsymbol{S} = 2\psi_{,\boldsymbol{C}} = 2\boldsymbol{P}^{-1}\varphi_{E,\boldsymbol{P}^{-T}\boldsymbol{C}\boldsymbol{P}^{-1}}\boldsymbol{P}^{-T}$ for the elastic constitutive relation. The restriction (9) then yields the simplified form $\boldsymbol{S}_E = 2\varphi_{E,\boldsymbol{C}_E}$ of $\boldsymbol{S} = 2\psi_{,\boldsymbol{C}}$, with $\boldsymbol{C}_E := \boldsymbol{P}^{-T}\boldsymbol{C}\boldsymbol{P}^{-1}$, $\boldsymbol{S}_E := \boldsymbol{P}\boldsymbol{S}\boldsymbol{P}^T$ and $\varphi_E = \varphi_E(\boldsymbol{C}_E)$. Further, since $\boldsymbol{P}^{-T}\boldsymbol{C}\boldsymbol{P}^{-1} = (\boldsymbol{F}\boldsymbol{P}^{-1})^T(\boldsymbol{F}\boldsymbol{P}^{-1})$, (9) also implies that the deformation measure

$$\boldsymbol{E} := \boldsymbol{F}\boldsymbol{P}^{-1}, \tag{10}$$

such that $\boldsymbol{C}_E = \boldsymbol{E}^T\boldsymbol{E}$, is directly associated with the elastic response of the material. Since the constitutive forms $\boldsymbol{S}_E = 2\varphi_{E,\boldsymbol{C}_E}$ and $\varphi_E = \varphi_E(\boldsymbol{C}_E)$ coincide with those commonly assumed in the usual formulation of hyperelastoplasticity as based on the elastoplastic decomposition $\boldsymbol{F} = \boldsymbol{F}_E\boldsymbol{F}_P$ of $\boldsymbol{F}$, we see that, at least from the material behaviour point of view, the assumption that $\boldsymbol{P}$ preserves the form of $2\psi_{,\boldsymbol{C}}$ lies behind the decomposition $\boldsymbol{F} = \boldsymbol{F}_E\boldsymbol{F}_P$. In this case, then, we may associate $\boldsymbol{P}$ with $\boldsymbol{F}_P$, and $\boldsymbol{E}$ with $\boldsymbol{F}_E$. On the other hand, in lieu of such an assumption, there is no reason from a constitutive point of view to relate $\boldsymbol{P}$ and $\boldsymbol{F}$ to each other via a relation of the form $\boldsymbol{E} = \boldsymbol{F}\boldsymbol{P}^{-1}$ at all.

Besides the reduced form (9) for ψ_E, the assumption that $\boldsymbol{P}$ does not affect the form of $(4)_2$, together with the form (8) for ψ, results in that

$$-\psi_{,\boldsymbol{P}} = 2\boldsymbol{C}_E\varphi_{E,\boldsymbol{C}_E}\boldsymbol{P}^{-T} - \psi_{P,\boldsymbol{P}} = \boldsymbol{M}\boldsymbol{P}^{-T} - \psi_{P,\boldsymbol{P}} \tag{11}$$

for the quantity $-\psi_{,\boldsymbol{P}}$ thermodynamically conjugate to $\dot{\boldsymbol{P}}$ in (5) via the chain rule, where

$$\boldsymbol{M} := 2\boldsymbol{C}_E\varphi_{E,\boldsymbol{C}_E} \tag{12}$$

represents the Mandel stress tensor (e.g., Mandel, 1972, 1974; see also Lubliner, 1986; Miehe, 1994). Substituting (11) into (5) then yields the form

† This condition on the form of the dependence of ψ_E on $\boldsymbol{P}$ represents a particular kind of material isomorphism (e.g., Noll, 1967, 1972), in particular, $\boldsymbol{P}$ then represents an elastic material isomorphism; see also Bertram (1993).

$$\begin{aligned}\delta_P &= \delta_P(\boldsymbol{C}, \boldsymbol{P}, \xi, \dot{\boldsymbol{C}}) \\ &= [\boldsymbol{M} - \psi_{P,\boldsymbol{P}}\boldsymbol{P}^T] \cdot \boldsymbol{L}_P - \psi_{P,\xi} \cdot \dot{\xi}\end{aligned} \tag{13}$$

for the plastic dissipation function δ_P, again with $\zeta = (\boldsymbol{P}, \xi)$, where we have introduced the usual form

$$\dot{\boldsymbol{P}} = \boldsymbol{L}_P \boldsymbol{P} \tag{14}$$

for $\dot{\boldsymbol{P}}$ in terms of the plastic velocity "gradient" $\boldsymbol{L}_P$. As such, (13) implies that the difference $\boldsymbol{M} - \psi_{P,\boldsymbol{P}}\boldsymbol{P}^T$ is thermodynamically conjugate to $\boldsymbol{L}_P$. In the context of an associated formulation of elastoplasticity as based on the notion of elastic range, or more specifically, a yield function, the form of (13) is then consistent with an interpretation of the quantity

$$\boldsymbol{X} := \psi_{P,\boldsymbol{P}}\boldsymbol{P}^T \tag{15}$$

as the centre of the elastic range of the material in the context of (8), i.e., the so-called back stress.

Many polycrystalline materials, in particular many metals, can be considered, even after plastically deforming (but before significant texture development occurs), to behave (at least approximately) elastically isotropically. Restricting ourselves to such materials in this work, we assume that ψ is isotropic with respect to the intermediate configuration. In particular, this implies that φ_E is an isotropic function of $\boldsymbol{C}_E$, in which case (12) reduces to†

$$\boldsymbol{M} = \varphi_{E,\ln \boldsymbol{U}_E}, \tag{16}$$

where $\ln \boldsymbol{U}_E$ represents the elastic logarithmic right stretch. This last form can be obtained from (12) in the context of elastic isotropy by, for example, working with the spectral form of these relations. A second consequence of the assumption that ψ is isotropic with respect to the intermediate configuration is the reduction of ψ_P to the form

$$\psi_P(\boldsymbol{P}, \xi) = \psi_P(\boldsymbol{C}_P, \xi) \tag{17}$$

(Svendsen, 1998b), where $\boldsymbol{C}_P := \boldsymbol{P}^T\boldsymbol{P}$ is the plastic right Cauchy–Green deformation tensor. This last reduction in the symmetric form

$$\boldsymbol{X} = 2\boldsymbol{P}\psi_{P,\boldsymbol{C}_P}\boldsymbol{P}^T \tag{18}$$

for $\boldsymbol{X}$ from (15). Consequently, in the current thermodynamic approach, both $\boldsymbol{M}$ and $\boldsymbol{X}$ are symmetric when ψ is isotropic with respect to the intermediate configuration, reducing δ_P in (13) to

$$\begin{aligned}\delta_P &= \delta_P(\boldsymbol{C}_E, \boldsymbol{P}, \xi, \dot{\boldsymbol{C}}) \\ &= [\boldsymbol{M} - \boldsymbol{X}] \cdot \boldsymbol{D}_P - \psi_{P,\xi} \cdot \dot{\xi},\end{aligned} \tag{19}$$

where

$$\boldsymbol{D}_P := \mathrm{sym}(\boldsymbol{L}_P) \tag{20}$$

represents the symmetric part of $\boldsymbol{L}_P$ as usual.

Lastly, note that the relation $\boldsymbol{K} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$ between the second Piola–Kirchhoff and Kirchhoff tensors, as well as $(4)_2$, (8), (9) and (12), lead to the form

† Note that $\ln \boldsymbol{U} = \frac{1}{2}\ln(\boldsymbol{C})$, and that $\ln(\boldsymbol{C})$ is an isotropic function of $\boldsymbol{C}$.

$$\boldsymbol{K} = 2\boldsymbol{F}\psi_{,\boldsymbol{C}}\boldsymbol{F}^{\mathrm{T}} = 2\boldsymbol{E}\varphi_{\mathrm{E},\boldsymbol{C}_{\mathrm{E}}}\boldsymbol{E}^{\mathrm{T}} = \boldsymbol{E}^{-\mathrm{T}}\boldsymbol{M}\boldsymbol{E}^{\mathrm{T}} \tag{21}$$

for $\boldsymbol{K}$ in terms of $\boldsymbol{M}$. By analogy, the form $\boldsymbol{A}$ of $\boldsymbol{X}$ with respect to the actual configuration is defined as

$$\boldsymbol{A} := \boldsymbol{E}^{-\mathrm{T}}\boldsymbol{X}\boldsymbol{E}^{\mathrm{T}}. \tag{22}$$

Since $\boldsymbol{M}$ and $\boldsymbol{C}_{\mathrm{E}}$ (or $\boldsymbol{U}_{\mathrm{E}}$) are coaxial in the case of elastic isotropy via (12), (21) reduces to

$$\boldsymbol{K} = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{M}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} \tag{23}$$

in this case via the polar decomposition

$$\boldsymbol{E} = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{U}_{\mathrm{E}} \tag{24}$$

of $\boldsymbol{E}$. On the other hand, $\boldsymbol{U}_{\mathrm{E}}$ and $\boldsymbol{X}$ are in general never coaxial; in the context of small elastic strain, however (22) does reduce to the form (23), as we shall see below.

3. ARMSTRONG–FREDERICK KINEMATIC HARDENING

To investigate the above formulation further, we consider in this section its specialization to the model of Armstrong–Frederick (1966) for non-linear kinematic hardening. The original form of this model was limited to small deformation, and for such deformations has been extended and formulated in a thermodynamic context by Chaboche and his collaborators (see, e.g., Chaboche, 1993). Various non-thermodynamic generalizations of this model to the large deformation case have been proposed on micromechanical (e.g., Dafalias, 1983, 1985; Loret, 1983; Aifantis, 1987, 1995) and/or phenomenological (e.g., Haupt and Tsakmakis, 1986, 1989; Svendsen and Tsakmakis, 1994) grounds. Phenomenologically, both cases can be represented with respect to the intermediate configuration by the classical Armstrong–Frederick rate form

$$\overset{*}{\boldsymbol{X}} = c\boldsymbol{D}_{\mathrm{P}} - \dot{z}\boldsymbol{X} \tag{25}$$

for the evolution of $\boldsymbol{X}$, again with respect to the intermediate configuration. Here, $\overset{*}{\boldsymbol{X}}$ represents some time derivative of $\boldsymbol{X}$, c represents the usual Armstrong–Frederick linear hardening parameter, and z the corresponding saturation variable. In the simplest case, z is given by

$$z = bs, \tag{26}$$

with b the classical Armstrong–Frederick saturation parameter, and s the plastic arc-length, i.e., $\dot{s} = |\boldsymbol{D}_{\mathrm{P}}|$.

As shown by Svendsen (1998a), one obtains from (8) and (18) the Armstrong–Frederick form (25) with respect to the intermediate configuration when ψ_{P} is given by

$$\psi_{\mathrm{P}}(\boldsymbol{C}_{\mathrm{P}}, \boldsymbol{Y}, \varepsilon) = \tfrac{1}{2}c[\boldsymbol{C}_{\mathrm{P}} \cdot \boldsymbol{Y} - \tfrac{1}{2}\ln(\det(\boldsymbol{C}_{\mathrm{P}}))] - h(\varepsilon), \tag{27}$$

where now $\xi = (\boldsymbol{Y}, \varepsilon)$. Here, $\boldsymbol{Y}$ is a strain-like internal variable associated with non-linear kinematic hardening determined by the evolution relation

$$\dot{\boldsymbol{Y}} = [\tfrac{1}{2}\boldsymbol{C}_{\mathrm{P}}^{-1} - \boldsymbol{Y}]\dot{z} \tag{28}$$

in the Armstrong–Frederick case. Further, ε is a strain-like internal variable associated with isotropic hardening. On the basis of (27), $\boldsymbol{X}$ then takes the explicit form

$$\boldsymbol{X} = c[\boldsymbol{PYP}^{\mathrm{T}} - \tfrac{1}{2}\boldsymbol{I}] \tag{29}$$

as a function $\boldsymbol{Y}$ from (18). In this case, (18) also yields the form

$$\overset{*}{\boldsymbol{X}} = \dot{\boldsymbol{X}} - \boldsymbol{L}_{\mathrm{P}}\boldsymbol{X} - \boldsymbol{X}\boldsymbol{L}_{\mathrm{P}}^{\mathrm{T}} \tag{30}$$

for $\overset{*}{\boldsymbol{X}}$ appearing in (25). The evolution relation (28) for $\boldsymbol{Y}$, and the particular form (27) for ψ_{P}, reduce δ_{P} to

$$\begin{aligned}\delta_{\mathrm{P}} &= \delta_{\mathrm{P}}(\boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Y}, \varepsilon, \dot{\boldsymbol{C}}) \\ &= [\boldsymbol{M} - \boldsymbol{X}] \cdot \boldsymbol{D}_{\mathrm{P}} + c[\tfrac{1}{2}\boldsymbol{C}_{\mathrm{P}} \cdot \boldsymbol{Y} - \tfrac{3}{4}]\dot{z} + \sigma\dot{\varepsilon}\end{aligned} \tag{31}$$

from (19), where we have now introduced the yield stress

$$\sigma := -\psi_{,\varepsilon} = h'. \tag{32}$$

With $c > 0$ and $\dot{z} > 0$, note that the condition $\boldsymbol{C}_{\mathrm{P}} \cdot \boldsymbol{Y} = \boldsymbol{I} \cdot \boldsymbol{PYP}^{\mathrm{T}} \geqslant \frac{3}{2}$ is, in the context of (26), sufficient to insure that the internal dissipation rate density is non-negative in all admissible inelastic processes. In particular, since the initial value $\boldsymbol{X}(0)$ of $\boldsymbol{X}$ is zero, we have $\boldsymbol{Y}(0) = \frac{1}{2}\boldsymbol{I}$ from (29), in which case $\boldsymbol{C}_{\mathrm{P}}(0) \cdot \boldsymbol{Y}(0) = \boldsymbol{I} \cdot \boldsymbol{P}(0)\boldsymbol{Y}(0)\boldsymbol{P}^{\mathrm{T}}(0) = \frac{1}{2}\boldsymbol{I} \cdot \boldsymbol{I} = \frac{3}{2}$. Initially, then, this condition is satisfied identically.

4. ASSOCIATED PLASTICITY FORMULATION AND MODEL SUMMARY

In the context of elastic isotropy, the above formulation can be embedded directly into the framework of associated plasticity (e.g., Lubliner, 1984). As is well-known, in this case, the elastic range of the material is given by a yield function

$$\phi = \phi(\boldsymbol{M} - \boldsymbol{X}, \sigma) \leqslant 0. \tag{33}$$

In view of (12), (15) and (32), we see that ϕ can also be expressed as a function of $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{Y}$ and ε. In addition, in this case, the loading function l appearing in (2) takes the special form $l(\boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Y}, \varepsilon, \dot{\boldsymbol{C}}) := \phi_{,\boldsymbol{C}} \cdot \dot{\boldsymbol{C}}$. Further, the plastic deformation rate $\boldsymbol{D}_{\mathrm{P}}$ is assumed to take the usual "normal" form

$$\boldsymbol{D}_{\mathrm{P}} = \lambda\phi_{,\boldsymbol{M}} \tag{34}$$

in the associated case, λ being the so-called plastic multiplier, with $\lambda \geqslant 0$ sufficient to insure $\delta_{\mathrm{P}} \geqslant 0$ when $\phi(\cdot, \varepsilon)$ is convex. As is also well-known, when this latter condition holds, the normality relation (34) is equivalent to the assumption that δ_{P} is maximal for $\boldsymbol{M}$ on the boundary of the elastic range, i.e., maximal for values of $\boldsymbol{M}$ satisfying $\phi(\boldsymbol{M} - \boldsymbol{X}, \sigma) = 0$. The associated formulation is completed by the so-called consistency condition

$$\dot{\phi} = 0 \tag{35}$$

during loading, requiring the state to remain on the boundary of the elastic range during (plastic) loading processes. As usual, this last relation determines λ.

As evident from (19), in the context of elastic isotropy, only the symmetric part $\boldsymbol{D}_{\mathrm{P}}$ of $\boldsymbol{L}_{\mathrm{P}}$ is constrained thermodynamically in the context of maximal plastic dissipation, motivating in particular its corresponding compatible potential-based constitutive form (34). This being the case, a constitutive form for the skew-symmetric part of

$$\boldsymbol{W}_{\mathrm{P}} := \mathrm{skw}(\boldsymbol{L}_{\mathrm{P}}) \tag{36}$$

of $\boldsymbol{L}_{\mathrm{P}}$, which is related to the "plastic spin" (i.e., as defined by, e.g., Dafalias, 1983, 1985), must in general be formulated on the basis of other considerations. Since it is not the purpose of this work to investigate true constitutive models for $\boldsymbol{W}_{\mathrm{P}}$ (e.g., Dafalias, 1985; Paulun and Pecherski, 1987, 1992; Aifantis, 1987, 1995), we focus on the cases in which it is either (1), set to zero (as done in, e.g., Eterovic and Bathe, 1990, or Weber and Anand, 1990), or, (2) determined in the framework of small elastic strain, in what follows.

To summarize, then, the model for hyperelastoplasticity with isotropic and Armstrong–Frederick kinematic hardening at large deformation formulated in this work consists of the evolution relations

$$\begin{aligned}
\dot{\boldsymbol{P}} &= \boldsymbol{L}_{\mathrm{P}}(\boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Y}, \varepsilon, \dot{\boldsymbol{C}})\boldsymbol{P} \\
\dot{\boldsymbol{Y}} &= b[\tfrac{1}{2}\boldsymbol{C}_{\mathrm{P}}^{-1} - \boldsymbol{Y}]|\boldsymbol{D}_{\mathrm{P}}|, \\
\dot{\varepsilon} &= \dot{\varepsilon}(\boldsymbol{C}, \boldsymbol{P}, \boldsymbol{Y}, \varepsilon, \dot{\boldsymbol{C}}), \\
\dot{\phi} &= 0,
\end{aligned} \tag{37}$$

during loading for the independent variables $\boldsymbol{P}$, $\boldsymbol{Y}$, ε, and λ, respectively, from (2) with $\zeta = (\boldsymbol{P}, \boldsymbol{Y}, \varepsilon)$, (14), (28), (26), and (35), with

$$\boldsymbol{L}_{\mathrm{P}} = \boldsymbol{D}_{\mathrm{P}} + \boldsymbol{W}_{\mathrm{P}} = \lambda \phi_{,\boldsymbol{M}} + \boldsymbol{W}_{\mathrm{P}} \tag{38}$$

from (20), (34) and (36), as well as the auxiliary relations

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{F}\boldsymbol{P}^{-1}, \\
\boldsymbol{M} &= \varphi_{\mathrm{E},\ln \boldsymbol{U}_{\mathrm{E}}}, \\
\boldsymbol{X} &= c[\boldsymbol{P}\boldsymbol{Y}\boldsymbol{P}^{\mathrm{T}} - \tfrac{1}{2}\boldsymbol{I}], \\
\sigma &= h'(\varepsilon), \\
0 &\geqslant \phi(\boldsymbol{M} - \boldsymbol{X}, \sigma)
\end{aligned} \tag{39}$$

from (10), (16), (29), (32), and (33), respectively.

This completes the model formulation. Our next task is the numerical integration of the evolution relations and formulation of the resulting algorithmic form of the model required, e.g., for the finite element implementation of the model.

5. NUMERICAL INTEGRATION

Following Eterovic and Bathe (1990), and Weber and Anand (1990), the hyperelastoplastic algorithm to be developed here is based on a direct implicit time integration of (37) over a time interval $[t_n, t_{n+1}]$. In particular, assuming $\boldsymbol{L}_{\mathrm{P}}$ is approximately constant in this interval, such integration of $(37)_1$ yields

$$\boldsymbol{P}_{n+1} = \exp(\tau \boldsymbol{L}_{\mathrm{P}})_{n+1} \boldsymbol{P}_n, \tag{40}$$

with

$$\tau_{n+1} := t_{n+1} - t_n \tag{41}$$

the corresponding time step, and

$$\exp(\boldsymbol{A}) := \sum_{n=1}^{\infty} \frac{1}{n!} \boldsymbol{A}^n \tag{42}$$

the usual exponential mapping. For notational simplicity, we neglect from now on the subscript $n+1$ on the corresponding algorithmic quantities when no confusion should arise. On the basis of the multiplicative relation $(39)_1$, (40) takes the alternative form

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{tr}} \exp(\tau \boldsymbol{L}_{\mathrm{P}})^{-1} \tag{43}$$

in terms of $\boldsymbol{E}$ and the quantity

$$\boldsymbol{E}_{\mathrm{tr}} := \boldsymbol{F}_{n+1,n} \boldsymbol{E}_n \tag{44}$$

representing the so-called trial elastic deformation, with

$$\boldsymbol{F}_{n+1,n} := \boldsymbol{F}_{n+1} \boldsymbol{F}_n^{-1} \tag{45}$$

the prescribed relative deformation. Because $\boldsymbol{M}$ is directly related to the elastic deformation via $(39)_2$, it turns out to be more convenient numerically to work with the form (43) of (40) in terms of $\boldsymbol{E}$; in the process, we replace $\boldsymbol{P}$ by $\boldsymbol{E}$ as an independent variable in the (algorithmic) formulation.

Next, we obtain the general algorithmic form for $(37)_2$ with $(39)_3$ associated with the thermodynamic formulation of Armstrong–Frederick kinematic hardening being considered in this work. To do this, it turns out to be more convenient to work algorithmically with the "push-forward"

$$\boldsymbol{Z} := \boldsymbol{P} \boldsymbol{Y} \boldsymbol{P}^{\mathrm{T}} \tag{46}$$

of $\boldsymbol{Y}$ to the intermediate configuration. With respect to $\boldsymbol{Z}$, $\boldsymbol{X}$ takes the form

$$\boldsymbol{X} = c[\boldsymbol{Z} - \tfrac{1}{2}\boldsymbol{I}] \tag{47}$$

via (29), with

$$(\boldsymbol{P}^{-1} \boldsymbol{Z} \boldsymbol{P}^{-\mathrm{T}})^{\cdot} = \dot{\boldsymbol{Y}} = b[\tfrac{1}{2}\boldsymbol{C}_{\mathrm{P}}^{-1} - \boldsymbol{Y}]|\boldsymbol{D}_{\mathrm{P}}| \tag{48}$$

and so

$$\overset{*}{\boldsymbol{Z}} = \dot{\boldsymbol{Z}} - \boldsymbol{L}_{\mathrm{P}} \boldsymbol{Z} - \boldsymbol{Z} \boldsymbol{L}_{\mathrm{P}}^{\mathrm{T}} = \boldsymbol{P} \dot{\boldsymbol{Z}} \boldsymbol{P}^{\mathrm{T}} = b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\boldsymbol{D}_{\mathrm{P}}|. \tag{49}$$

In view of (46)–(49), then, backward-Euler integration of $(37)_2$ leads to the following algorithmic form

$$\boldsymbol{Z}_n = \exp(\tau \boldsymbol{L}_{\mathrm{P}})^{-1} \{\boldsymbol{Z} - b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\tau \boldsymbol{D}_{\mathrm{P}}|\} \exp(\tau \boldsymbol{L}_{\mathrm{P}})^{-\mathrm{T}} \tag{50}$$

for this evolution relation in terms of $\boldsymbol{Z}$ with respect to the intermediate configuration.

Together with (43) and (50), then, backward-Euler integration of the remaining evolution relations in (37) yields the set

$$\begin{aligned}
&\boldsymbol{E}_{\mathrm{tr}} = \boldsymbol{E} \exp(\tau \boldsymbol{L}_{\mathrm{P}}), \\
&\boldsymbol{Z}_n = \exp(\tau \boldsymbol{L}_{\mathrm{P}})^{-1} \{\boldsymbol{Z} - b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\tau \boldsymbol{D}_{\mathrm{P}}|\} \exp(\tau \boldsymbol{L}_{\mathrm{P}})^{-\mathrm{T}}, \\
&\varepsilon_n = \varepsilon - \tau \dot{\varepsilon}, \\
&0 = \phi,
\end{aligned} \tag{51}$$

of non-linear algebraic relations for the independent variables $\boldsymbol{E}$, $\boldsymbol{Z}$, ε, and

$$\gamma := \tau\lambda, \tag{52}$$

respectively, of the algorithmic form of the model. The initial values of the inelastic variables of the model are as usual those at time $t = t_n$, and we have used the fact that $\phi_n = 0$ in writing $(51)_4$. With the help of (39) and (46), the solution of (51) yields $\boldsymbol{M}$, $\boldsymbol{X}$, σ and γ, all at $t = t_{n+1}$. In turn, the Kirchhoff stress $\boldsymbol{K}$ and its back-stress counterpart $\boldsymbol{A}$ with respect to the current configuration are obtained from these via (23) and (22), respectively.

6. SMALL INCREMENTAL PLASTIC DEFORMATION AND SMALL ELASTIC STRAIN

The algorithmic system (51), together with (39) and (46), describes elastoplastic material behaviour with non-linear isotropic, and Armstrong–Frederick kinematic, hardening for arbitrarily large elastic and plastic deformations. In this general form, however, it cannot be solved exactly, a fact due in essence to the exponential term $\exp(\tau\boldsymbol{L}_P)$ appearing in $(51)_{1,2}$. To proceed further, then, we follow Eterovic and Bathe (1990) in assuming† that the magnitude

$$\varepsilon_P := |\tau\boldsymbol{L}_P| \tag{53}$$

of the incremental plastic deformation $\tau\boldsymbol{L}_P$ is much less than one. Since $|\tau\boldsymbol{L}_P| = |\tau\boldsymbol{D}_P| + |\tau\boldsymbol{W}_P|$, this requires both $|\tau\boldsymbol{D}_P|$ and $|\tau\boldsymbol{W}_P|$ to be so as well. On this basis, the expansion‡

$$\exp(\tau\boldsymbol{L}_P) = \boldsymbol{I} + \tau\boldsymbol{L}_P + \boldsymbol{O}(\varepsilon_P^2) \tag{54}$$

from (42) leads to the $\boldsymbol{O}(\varepsilon_P^2)$-approximations

$$\begin{aligned} \boldsymbol{E}_{tr} &= \boldsymbol{E}[\boldsymbol{I} + \tau\boldsymbol{L}_P] + \boldsymbol{O}(\varepsilon_P^2), \\ \boldsymbol{Z}_n &= \boldsymbol{Z} - b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\tau\boldsymbol{D}_P| - [\tau\boldsymbol{W}_P, \boldsymbol{Z}] - \langle\tau\boldsymbol{D}_P, \boldsymbol{Z}\rangle + \boldsymbol{O}(\varepsilon_P^2), \end{aligned} \tag{55}$$

to $(51)_{1,2}$. Here, $[\boldsymbol{A}, \boldsymbol{B}] := \boldsymbol{AB} - \boldsymbol{BA}$ represent the Lie bracket of any two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, and $\langle\boldsymbol{A}, \boldsymbol{B}\rangle := \boldsymbol{AB} + \boldsymbol{BA}$ their Jacobi bracket. With $(51)_{1,2}$ replaced by their corresponding approximations $(55)_{1,2}$ in (51), the resulting system becomes numerically tractable (subject of course to having constitutive forms for quantities such as ϕ and $\boldsymbol{W}_P$).

To incorporate further simplifications into this system, it is useful to use the polar decomposition (24) of $\boldsymbol{E}$ to "split" $(55)_1$ into the algorithmic form

$$\boldsymbol{C}_{Etr} = \boldsymbol{C}_E - [\tau\boldsymbol{W}_P, \boldsymbol{C}_E] + \langle\tau\boldsymbol{D}_P, \boldsymbol{C}_E\rangle + \boldsymbol{O}(\varepsilon_P^2) \tag{56}$$

for $\boldsymbol{C}_E$ (and so $\boldsymbol{U}_E$), as well as that

$$\boldsymbol{R}_E^T\boldsymbol{R}_{Etr} = \boldsymbol{U}_E[\boldsymbol{I} + \tau\boldsymbol{L}_P]\boldsymbol{U}_{Etr}^{-1} + \boldsymbol{O}(\varepsilon_P^2) \tag{57}$$

for $\boldsymbol{R}_E$, both as in $(55)_1$ to $\boldsymbol{O}(\varepsilon_P^2)$. That $\boldsymbol{R}_E^T\boldsymbol{R}_{Etr}$ (and so $\boldsymbol{R}_E$) as given by (57) actually represents a rotation [i.e., to $\boldsymbol{O}(\varepsilon_P^2)$] can be shown using the definition of a rotation and (56). In this case, (51) is replaced by the system

† To be more specific, Eterovic and Bathe (1990) assumed $|\tau\boldsymbol{D}_P| \ll 1$ and $\boldsymbol{W}_P = \boldsymbol{0}$.

‡ Note that this expansion holds exactly when $\boldsymbol{L}_P$ is nilpotent of index 2, i.e., $\boldsymbol{L}_P^2 = \boldsymbol{0}$ but $\boldsymbol{L}_P \neq \boldsymbol{0}$. This would be the case, for example, if $\boldsymbol{P}$ represented a simple shear.

$$\begin{aligned} \boldsymbol{C}_{\mathrm{Etr}} &= \boldsymbol{C}_{\mathrm{E}} - [\tau \boldsymbol{W}_{\mathrm{P}}, \boldsymbol{C}_{\mathrm{E}}] + \langle \tau \boldsymbol{D}_{\mathrm{P}}, \boldsymbol{C}_{\mathrm{E}} \rangle, \\ \boldsymbol{Z}_n &= \boldsymbol{Z} - b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\tau \boldsymbol{D}_{\mathrm{P}}| - [\tau \boldsymbol{W}_{\mathrm{P}}, \boldsymbol{Z}] - \langle \tau \boldsymbol{D}_{\mathrm{P}}, \boldsymbol{Z} \rangle, \\ \varepsilon_n &= \varepsilon - \tau \dot{\varepsilon}, \\ 0 &= \phi, \end{aligned} \tag{58}$$

of non-linear relations to $\boldsymbol{O}(\varepsilon_{\mathrm{P}}^2)$ involving the variables $\boldsymbol{C}_{\mathrm{E}}$, $\boldsymbol{Z}$, ε and γ. Solution of this last system yields then $\boldsymbol{C}_{\mathrm{E}}$ (and so $\boldsymbol{U}_{\mathrm{E}}$) and the other (plastic) deformation quantities. In turn, $\boldsymbol{U}_{\mathrm{E}}$ can then be used to solve (57) for $\boldsymbol{R}_{\mathrm{E}}$, and so obtain $\boldsymbol{E}$. The rest is the same as before.

A further simplification of the model arises when we restrict attention to materials whose elastic ranges are "small" in the sense that the magnitudes of the yield stress and other internal variables are such that the boundary of the elastic range is reached before (at least a part of) the magnitude of elastic strain becomes "large". The classic example of such materials is of course metals. In the current isotropic hyperelastic context, the strain measure involved is the elastic right logarithmic stretch $\ln \boldsymbol{U}_{\mathrm{E}}$, as appears in (16). More specifically, in the case of elastic isotropy and von Mises plasticity, for example, note that only $\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})$ would necessarily remain "small", i.e., $\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})$, and so $|\ln \boldsymbol{U}_{\mathrm{E}}| = |\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})| + |\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})|$, could still be "large" in this case. On the other hand, if yielding also depends on volumetric processes, e.g., damage, it may be that $\ln \boldsymbol{U}_{\mathrm{E}}$ itself would remain "small". For simplicity, however, we restrict ourselves to von Mises plasticity in the rest of this work. As such, it is useful to work with the particular isotropic form

$$\varphi_{\mathrm{E}} = \varphi_{\mathrm{E}}(\ln \boldsymbol{U}_{\mathrm{E}}) = \varphi_{\mathrm{E}}(\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}}), \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})) = \varphi_{\mathrm{E}}(I_{\mathrm{E}1}, I_{\mathrm{E}2}, I_{\mathrm{E}3}) \tag{59}$$

for φ_{E} depending explicitly on the independent invariants

$$\begin{aligned} I_{\mathrm{E}1} &:= \mathrm{tr}(\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})) = \mathrm{tr}(\ln \boldsymbol{U}_{\mathrm{E}}), \\ I_{\mathrm{E}2} &:= \mathrm{tr}(\mathrm{sq}(\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}))), \\ I_{\mathrm{E}3} &:= \mathrm{tr}(\mathrm{cu}(\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}))), \end{aligned} \tag{60}$$

of $\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})$ and $\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})$. Substituting this into (16) yields then the particular isotropic form

$$\boldsymbol{M} = \varphi_{\mathrm{E},\ln \boldsymbol{U}_{\mathrm{E}}} = \varphi_{\mathrm{E},I_{\mathrm{E}1}} \boldsymbol{I} + 2\varphi_{\mathrm{E},I_{\mathrm{E}2}} \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) + 3\varphi_{\mathrm{E},I_{\mathrm{E}3}} \mathrm{dev}(\mathrm{sq}(\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}))), \tag{61}$$

for $\boldsymbol{M}$ in terms of $\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})$ and $\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})$.

In effect, the condition of small elastic strain represents an implicit restriction on the magnitude of the yield stress σ as a function of the magnitudes of the elastic constants and the inelastic variables such as $\boldsymbol{X}$. Roughly speaking, this condition will hold when the magnitude of the yield stress is and remains small compared to that of the elastic constants. For example, in the simple case of von Mises plasticity without kinematic hardening, we have the condition

$$\sigma^2 = \tfrac{3}{2}|\mathrm{dev}(\boldsymbol{M})|^2 \tag{62}$$

for the boundary of the elastic range from the corresponding yield relation. Expanding now $\boldsymbol{M}$ as given in (61) about the state $\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) = \boldsymbol{0}$, we obtain

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{M}|_{\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})=\boldsymbol{0}} + \boldsymbol{M}_{,\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})}|_{\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})=\boldsymbol{0}} [\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})] + \boldsymbol{O}(|\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})|^2) \\ &= \varphi_{\mathrm{E},I_{\mathrm{E}1}}|_{\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})=\boldsymbol{0}} \boldsymbol{I} + 2\varphi_{\mathrm{E},I_{\mathrm{E}2}}|_{\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})=\boldsymbol{0}} \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) + \boldsymbol{O}(|\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}})|^2). \end{aligned} \tag{63}$$

Here, $\varphi_{E,I_{E1}}|_{\mathrm{dev}(\ln U_E)=0}$ and $\varphi_{E,I_{E2}}|_{\mathrm{dev}(\ln U_E)=0}$ are of course (still) functions of $\mathrm{sph}(\ln U_E)$, whose magnitude could in general be "large". Substituting (63) into (62) then yields the condition

$$\frac{\sigma^2}{6(\varphi_{E,I_{E2}}|_{\mathrm{dev}(\ln U_E)=0})^2} = |\mathrm{dev}(\ln U_E)|^2 + O(|\mathrm{dev}(\ln U_E)|^3) \gtrsim |\mathrm{dev}(\ln U_E)|^2 \tag{64}$$

on the maximum magnitude of $\mathrm{dev}(\ln U_E)$ attainable in the von Mises context. With $\sigma \sim 1$ GPa, and $\varphi_{E,I_{E2}}|_{\mathrm{dev}(\ln U_E)=0} \sim 100$ GPa, for example (i.e., typical order of magnitude values for metals), (64) would imply $|\mathrm{dev}(\ln U_E)| \lesssim 10^{-2}$.

Restricting ourselves now to this case, i.e., when

$$\varepsilon_E := |\mathrm{dev}(\ln U_E)| \ll 1 \tag{65}$$

holds, it is physically reasonable to work in the algorithm with the approximate, linearized form

$$M = \varphi_{E,I_{E1}}|_{\mathrm{dev}(\ln U_E)=0} I + 2\varphi_{E,I_{E2}}|_{\mathrm{dev}(\ln U_E)=0}\,\mathrm{dev}(\ln U_E) = C_E[\ln U_E] \tag{66}$$

from $(63)_2$ of the isotropic hyperelastic constitutive relation (61), where

$$C_E := \varphi_{E,\ln U_E \ln U_E}|_{\mathrm{dev}(\ln U_E)=0} = \lambda(I \otimes I) + 2\mu I = \kappa(I \otimes I) + 2\mu\,\mathrm{dev} \tag{67}$$

(recall that $\mathrm{dev} = I - \mathrm{sph}$; see Section 1) represents the isotropic elasticity tensor at $\mathrm{dev}(\ln U_E) = 0$,

$$\kappa := \varphi_{EI_{E1}}|_{\mathrm{dev}(\ln U_E)=0} = \lambda + \tfrac{2}{3}\mu \quad \text{and} \quad \mu := \varphi_{E,I_{E2}}|_{\mathrm{dev}(\ln U_E)=0} \tag{68}$$

being the corresponding bulk and shear moduli, respectively, with respect to this state. Additional simplifications of the algorithm can be obtained on the basis of the relation

$$U_E = \exp(\ln U_E) = \exp(\mathrm{sph}(\ln U_E))\exp(\mathrm{dev}(\ln U_E)) = e^{\mathrm{tr}(\ln U_E)}[I + \mathrm{dev}(\ln U_E) + O(\varepsilon_E^2)] \tag{69}$$

via (42). In particular, we obtain from this the approximation

$$\begin{aligned} A &= E^{-T} X E^T \\ &= R_E X R_E^T + O(\varepsilon_E) \end{aligned} \tag{70}$$

for the current configuration form A of the back stress given in (22). In addition, the multiplicative decomposition

$$F = EP \tag{71}$$

of F from (10) reduces to the form

$$F = R_E R_P e^{\mathrm{tr}(\ln U_E)} U_P + O(\varepsilon_E) \tag{72}$$

via (69) and the polar decomposition

$$P = R_P U_P \tag{73}$$

of P. The result (72), when combined with the uniqueness of the polar decomposition of F, yields the approximations

$$\boldsymbol{R} = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{R}_{\mathrm{P}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}),$$
$$\boldsymbol{U} = e^{\mathrm{tr}(\ln \boldsymbol{U}_{\mathrm{E}})}\boldsymbol{U}_{\mathrm{P}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \tag{74}$$

to $\boldsymbol{O}(\varepsilon_{\mathrm{E}})$, representing generalizations of similar relations obtained by Bammann and Johnson (1987) to the case when $|\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})|$ may be large.

Finally, consider the algorithmic elastoplastic relations (56) and (57) for $\boldsymbol{C}_{\mathrm{E}}$ and $\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}\boldsymbol{R}_{\mathrm{Etr}}$, respectively. In particular, on the basis of (69), the first of these reduces to the pair of relations

$$\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{Etr}}) = \mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}}) + \boldsymbol{O}(\varepsilon_{\mathrm{EP}}^2),$$
$$\mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{Etr}}) = \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) + \tau\boldsymbol{D}_{\mathrm{P}} + \boldsymbol{O}(\varepsilon_{\mathrm{EP}}^2), \tag{75}$$

for the elastic logarithmic right stretch with $\mathrm{tr}(\boldsymbol{D}_{\mathrm{P}}) = 0$ in the context of associated von Mises plasticity, where $\varepsilon_{\mathrm{EP}} := \sqrt{\varepsilon_{\mathrm{E}}\varepsilon_{\mathrm{P}}}$. Likewise, the results (69) and (75) lead to the approximation

$$\begin{aligned}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}\boldsymbol{R}_{\mathrm{Etr}} &= \boldsymbol{I} + \tau\boldsymbol{W}_{\mathrm{P}} + \tau\boldsymbol{D}_{\mathrm{P}} + \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) - \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{Etr}}) + \boldsymbol{O}(\varepsilon_{\mathrm{EP}}^2)\\ &= \boldsymbol{I} + \tau\boldsymbol{W}_{\mathrm{P}} + \boldsymbol{O}(\varepsilon_{\mathrm{EP}}^2),\end{aligned} \tag{76}$$

to (57) for the relative elastic rotation $\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}\boldsymbol{R}_{\mathrm{Etr}}$ to the same order. Together with this last result, $(74)_1$ implies that $\boldsymbol{R}_{\mathrm{E}}$, $\boldsymbol{R}_{\mathrm{P}}$ and $\tau\boldsymbol{W}_{\mathrm{P}}$, are, to $\boldsymbol{O}(\varepsilon_{\mathrm{E}})$, not all independent.

In the context of the assumption that ψ is isotropic with respect to the intermediate configuration, we see in particular that the material isomorphism condition (9) on $\boldsymbol{P}$ reduces to $\psi_{\mathrm{E}}(\boldsymbol{C}, \boldsymbol{P}) = \varphi_{\mathrm{E}}(\boldsymbol{U}_{\mathrm{P}}^{-1}\boldsymbol{C}\boldsymbol{U}_{\mathrm{P}}^{-1})$. In addition, as already discussed above, we have then $\psi_{\mathrm{P}}(\boldsymbol{P}, \xi) = \psi_{\mathrm{P}}(\boldsymbol{C}_{\mathrm{P}}, \xi)$ in (17). Consequently, ψ is independent of $\boldsymbol{R}_{\mathrm{P}}$ in this case. Since $\boldsymbol{R}_{\mathrm{P}}$ is not otherwise a part of the model, it is then in particular free to choose, the simplest choice being of course $\boldsymbol{R}_{\mathrm{P}} = \boldsymbol{I}$. The consequences of this and other aspects of the model will be investigated further in Section 8 in the context of simple shear.

7. SUMMARY OF ALGORITHM

In the context of backward-Euler integration, small incremental plastic deformation and small deviatoric elastic strain, then, the original system (37) of model evolution relations reduces to the algorithmic form

$$\begin{aligned}\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{Etr}}) &= \mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}}),\\ \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{Etr}}) &= \mathrm{dev}(\ln \boldsymbol{U}_{\mathrm{E}}) + \tau\boldsymbol{D}_{\mathrm{P}},\\ \boldsymbol{Z}_n &= \boldsymbol{Z} - b[\tfrac{1}{2}\boldsymbol{I} - \boldsymbol{Z}]|\tau\boldsymbol{D}_{\mathrm{P}}| - [\tau\boldsymbol{W}_{\mathrm{P}}, \boldsymbol{Z}] - \langle\tau\boldsymbol{D}_{\mathrm{P}}, \boldsymbol{Z}\rangle,\\ \varepsilon_n &= \varepsilon - \tau\dot{\varepsilon},\\ 0 &= \phi,\end{aligned} \tag{77}$$

via (58) and (75) for the variables $\ln \boldsymbol{U}_{\mathrm{E}}$, $\boldsymbol{Z}$, ε, and γ, respectively, with the relations for $\ln \boldsymbol{U}_{\mathrm{E}}$ holding to $\boldsymbol{O}(\varepsilon_{\mathrm{EP}}^2)$. In this system appears the Mandel $\boldsymbol{M}$, back $\boldsymbol{X}$, and yield σ, stresses, given by

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{C}_{\mathrm{E}}[\ln \boldsymbol{U}_{\mathrm{E}}],\\ \boldsymbol{X} &= c[\boldsymbol{Z} - \tfrac{1}{2}\boldsymbol{I}],\\ \sigma &= h'(\varepsilon),\end{aligned} \tag{78}$$

from (66), (47), and (32), respectively, where $(78)_1$ holds to $\boldsymbol{O}(\varepsilon_E^2)$. The trial values for the elastic quantities utilized in the predictor-corrector approach are given as usual by

$$\begin{aligned} \boldsymbol{R}_{Etr} &= \boldsymbol{R}(\boldsymbol{E}_{tr}), \\ ln\, \boldsymbol{U}_{Etr} &:= \tfrac{1}{2}\ln(\boldsymbol{E}_{tr}^T \boldsymbol{E}_{tr}), \end{aligned} \tag{79}$$

via (44). Finally, as discussed at the end of the last section, in the case that ψ is isotropic with respect to the intermediate configuration, it is natural to assume $\boldsymbol{R}_P = \boldsymbol{I}$; from this, we have

$$\boldsymbol{R}_E = \boldsymbol{R} \tag{80}$$

from $(74)_1$ to $\boldsymbol{O}(\varepsilon_E)$, and so

$$\tau \boldsymbol{W}_P = \boldsymbol{R}^T \boldsymbol{R}_{Etr} - \boldsymbol{I} \tag{81}$$

from (76) to $\boldsymbol{O}(\varepsilon_{EP}^2)$.

Solution of (77) with (78)–(81) yields the current values of $\boldsymbol{M}$, $\boldsymbol{X}$, σ and γ. These first two then determine the corresponding Kirchhoff stress $\boldsymbol{K}$ and back stress $\boldsymbol{A}$, with respect to the actual configuration via

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{R}_E \boldsymbol{M} \boldsymbol{R}_E^T, \\ \boldsymbol{A} &= \boldsymbol{R}_E \boldsymbol{X} \boldsymbol{R}_E^T, \end{aligned} \tag{82}$$

[the latter to $\boldsymbol{O}(\varepsilon_E)$] from (23) and (70), respectively. The next step is to investigate the behaviour of the current model, as well as some existing thermodynamic formulations for kinematic hardening, in the context of simple shear. Before we do this, however, we turn first to a brief review of these existing formulations and a comparison of these with the current formulation.

8. COMPARISON WITH PREVIOUS FORMULATIONS

The approach to the formulation of kinematic hardening taken in the current work is conceptually consistent with the first approach of Dogui and Sidoroff (1985) in the sense that ψ_P is assumed to depend explicitly on $\boldsymbol{P}$ (i.e., on $\boldsymbol{F}_P$ in their case). Indeed, in the context of isotropic material behaviour, they assumed the isotropic form

$$\psi_P = \psi_P(\boldsymbol{B}_P) \tag{83}$$

of ψ_P from the start, and so obtained that

$$\boldsymbol{X} = 2\psi_{P,\boldsymbol{B}_P} \boldsymbol{B}_P \tag{84}$$

for $\boldsymbol{X}$ as a function of the plastic form $\boldsymbol{B}_P$ of the left Cauchy–Green deformation $\boldsymbol{B} := \boldsymbol{F}\boldsymbol{F}^T$. As discussed in more detail elsewhere (Svendsen, 1998b), such a constitutive form for ψ_P can be obtained in the current formulation when (1), ψ is isotropic with respect to both the reference and intermediate configurations, and (2), ψ_P is independent of ξ. Because of this last assumption, their formulation does not encompass Armstrong–Frederick-type kinematic hardening. A related, more recent formulation incorporating a model for large-deformation Armstrong–Frederick-type kinematic hardening can be found in Haupt (1995). He assumed the isotropic form

$$\psi_P = \psi_P(\boldsymbol{B}_P, \boldsymbol{Y}) = \tfrac{1}{2}c(\boldsymbol{A}_P - \boldsymbol{Y})\cdot(\boldsymbol{A}_P - \boldsymbol{Y}) \tag{85}$$

for ψ_P, with $\boldsymbol{A}_P := \frac{1}{2}(\boldsymbol{I} - \boldsymbol{B}_P^{-1})$ the plastic Almansi strain tensor, and $\boldsymbol{Y}$ a strain-like internal variable accounting for non-linear hardening, analogous to the internal variable $\boldsymbol{Y}$ used in the current formulation. Rather than obtaining a relation of the form (15), however, he defined the back stress as

$$\boldsymbol{X} := [1 + 2(\boldsymbol{A}_P - \boldsymbol{Y})]\psi_{P,\boldsymbol{A}_P - \boldsymbol{Y}}, \tag{86}$$

a form motivated by that of the Mandel stress $\boldsymbol{M}$ as given in (12). This was also done by Tsakmakis (1996), as we discussed below.

In a second approach also discussed by Dogui and Sidoroff (1985), ψ_P is assumed to be independent of $\boldsymbol{F}_P$; such an approach (at least tacitly) assumed in the thermodynamical formulation of, e.g., Chaboche (1993) (when generalized to large deformations), or that of Tsakmakis (1996). In these cases, one usually works with the simple "quadratic" form

$$\psi_P(\boldsymbol{\alpha}, \varepsilon) = \tfrac{1}{2}c\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} - h(\varepsilon) \tag{87}$$

for ψ_P in terms of the strain-like internal variable $\boldsymbol{\alpha}$, whose evolution is governed by the Armstrong–Frederick form

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{D}_P - \dot{z}\boldsymbol{\alpha} \tag{88}$$

(i.e., neglecting static recovery) generalized to the intermediate configuration here. Finally, he assumes that the back stress is thermodynamically conjugate to $\boldsymbol{\alpha}$, i.e.,

$$\boldsymbol{X} := \psi_{,\boldsymbol{\alpha}} = c\boldsymbol{\alpha}. \tag{89}$$

On the other hand, Tsakmakis (1996) defined the back stress as

$$\boldsymbol{X} := (1 + 2\boldsymbol{\alpha})\psi_{,\boldsymbol{\alpha}}, \tag{90}$$

again analogous to the structure of the Mandel stress, as already mentioned above. In addition, he assumes that $\boldsymbol{\alpha}$ evolves in an Armstrong–Frederick fashion relative to the flow of $\boldsymbol{F}_P$, i.e., according to

$$\boldsymbol{F}_P(\boldsymbol{F}_P^{-1}\boldsymbol{\alpha}\boldsymbol{F}_P^{-T})^{\cdot}\boldsymbol{F}_P^T = \boldsymbol{D}_P - \dot{z}\boldsymbol{\alpha}, \tag{91}$$

rather than relative to the intermediate configuration directly via the form (88).

In the simple shear comparison to follow, we restrict ourselves to the generalization of the Chaboche (1993) approach to the intermediate configuration, as embodied by (88) and (89). In particular, backward-Euler integration of the former relation then results in the algorithmic relations

$$\begin{aligned} \boldsymbol{\alpha}_n &= (1 + b|\tau\boldsymbol{D}_P|)\boldsymbol{\alpha} - \tau\boldsymbol{D}_P, \\ \boldsymbol{X} &= c\boldsymbol{\alpha}, \end{aligned} \tag{92}$$

to determine $\boldsymbol{X}$ in the context of this class of models. In particular, these compare to $(77)_3$ and $(78)_2$, respectively, in the current formulation. All other aspects of the two formulations are the same.

9. SIMPLE SHEAR COMPARISON

In this section, we investigate the behaviour of the current formulation as embodied by (77) with (78)–(81), in the standard context of simple shear. We do this as well for the formulation following from the Chaboche (1993) approach generalized to the intermediate configuration, in which $(77)_3$ and $(78)_2$ are replaced by $(92)_{1,2}$, respectively, in the formulation. In the process, we focus in particular on the effect of $\boldsymbol{W}_P$ or $\boldsymbol{R}_P$ on this behaviour. To do this, we solve these two algorithmic formulations using the predictor-corrector approach for the case of two-dimensional simple shear, where the input is a shear of 500% in the x_1 direction. In the calculation, we assume a value for Young's modulus of 210 GPa, and one for Poisson's ratio of 0.3, these being typical of steels. To keep things simple, we neglect isotropic hardening, and so work with a constant yield stress σ of 0.5 GPa (again typical of steels). Finally, again for simplicity, we neglect saturation, focusing as such solely on linear kinematic hardening with $c = 1$ GPa, in what follows.

We begin with the Chaboche case, described algorithmically by the system $(77)_{1,2,5}$ and $(92)_1$ in this case, with $(78)_1$, (79) and $(92)_2$. Consider first the case $\boldsymbol{W}_P = \boldsymbol{0}$, which one finds explicitly or tacitly in, e.g., Eterovic and Bathe (1990), Weber and Anand (1990), Simo (1992), Zavaliangos and Anand (1993), and many others. In this case, (76) reduces to

$$\begin{aligned} \boldsymbol{R}_E &= \boldsymbol{R}_{Etr}, \\ \boldsymbol{R}_P &= \boldsymbol{R}_E^T \boldsymbol{R}, \end{aligned} \tag{93}$$

[to $\boldsymbol{O}(\varepsilon_E)$]. As such, $\boldsymbol{R}_P$ is determined by $\boldsymbol{R}$ and $\boldsymbol{R}_{Etr}$. The results for the 11, 22 and 12 components of $\boldsymbol{K}$ and $\boldsymbol{A}$ in this case as obtained from (82) are displayed in Fig 1(a). As shown there, one obtains oscillatory stresses in this case; in fact, these hyperelastoplastic results correspond exactly with the case of Jaumann-hypoelastoplasticity with linear kinematic hardening (e.g., Nagtegaal and DeJong, 1981), shown for comparison in Fig. 1(b). That neglecting the "plastic spin" $\boldsymbol{W}_P$ leads to such behaviour has long been known in the hypoelastoplastic context (e.g., Dafalias, 1985; Tvergaard and Van der Giessen, 1991); the results in Fig. 1(a) confirm this in the corresponding hyperelastoplastic case.

Less well-known are the results obtained by assuming $\boldsymbol{R}_P = \boldsymbol{I}$ instead of $\boldsymbol{W}_P = \boldsymbol{0}$, in which case we have

$$\begin{aligned} \boldsymbol{R}_E &= \boldsymbol{R}, \\ \tau \boldsymbol{W}_P &= \boldsymbol{R}_E^T \boldsymbol{R}_{Etr} - \boldsymbol{I}, \end{aligned} \tag{94}$$

again from (76). Here, then, $\tau \boldsymbol{W}_P$ is determined by $\boldsymbol{R}$ and $\boldsymbol{R}_{Etr}$. The results for the 11, 22 and 12 components of $\boldsymbol{K}$ and $\boldsymbol{A}$ in this case, again obtained from (82), are shown in Fig. 2(a). Analogous to the previous case, these results for $\boldsymbol{K}$ and $\boldsymbol{A}$ correspond exactly to those obtained for the hypoelastoplastic model for linear kinematic hardening based on the so-called Green–Naghdi rate; the analogous results for this latter case are shown in Fig. 2(b). This rate can be found in Truesdell and Noll (1965, 1992, Section 36), Green and Naghdi (1965), and was incorporated into the context of modern or "generalized" hypoelasticity by Green and McInnis (1967). More recently, it has been investigated by a number of authors, in particular Dienes (1979, 1986) as a physically more reasonable model than the Jaumann-based form in the sense that no stress oscillations are observed under (monotonic) simple shear. In the context of hypoelastoplasticity and hypoviscoplasticity, it has been utilized and discussed, e.g., by Bammann (1990), and more recently by Arndt *et al.* (1997) in the context of a hyperelastoplastic model for ductile damage and crack growth in metals.

The correspondence in these two cases with Jaumann- or Green–Naghdi-based hypoelastoplasticity with linear kinematic hardening is the result of two factors: (1), small elastic strains, and (2), the correspondence of the evolution relations for the back stress. Indeed, Jaumann- and Green–Naghdi-hypoelasticity agree with the corresponding linearized hyperelastic relation $(78)_1$ based on $\ln \boldsymbol{U}_E$ up to about 100% shear in the case of simple shear.

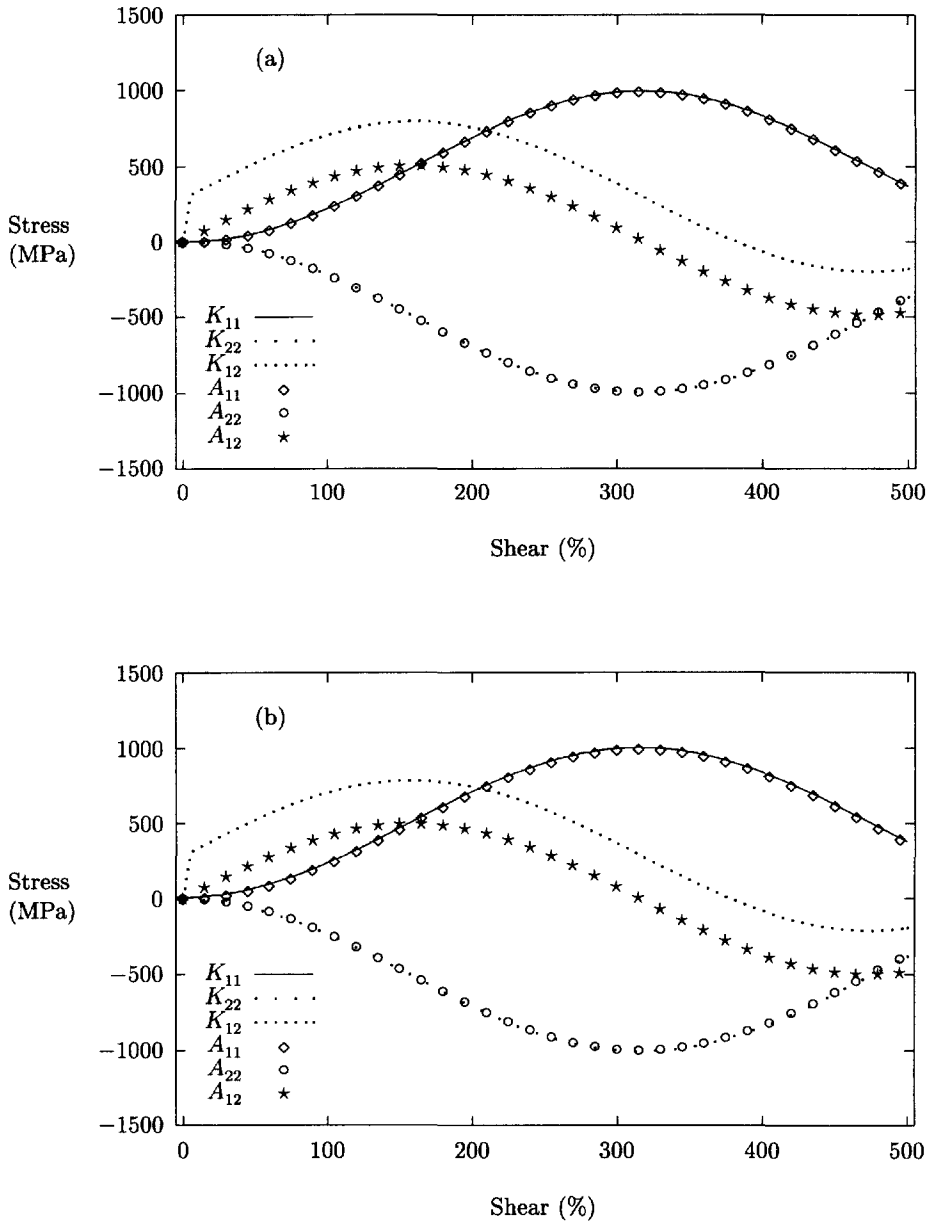


Fig. 1. Comparison of results for the 11, 22 and 12 components of the Kirchhoff stress $\boldsymbol{K}$ and current back stress $\boldsymbol{A}$ for 500% shear in the x_1 direction. Results in part (a) are based on solution of the system $(77)_{1,2,5}$ and $(92)_1$, with $(78)_1$, (79), $(92)_2$, and $\boldsymbol{W}_P = \boldsymbol{0}$. Those in part (b) are from the hypoelastoplastic Jaumann model. See text for more details.

The correspondence of the evolution relations for the back stress in each case can be deduced from the result

$$\begin{aligned}\dot{\boldsymbol{A}} - [\boldsymbol{W} - \boldsymbol{R}_E \boldsymbol{W}_P \boldsymbol{R}_E^T, \boldsymbol{A}] &= \boldsymbol{R}_E \dot{\boldsymbol{X}} \boldsymbol{R}_E^T + \boldsymbol{O}(\varepsilon_E) \\ &= c\boldsymbol{R}_E \boldsymbol{D}_P \boldsymbol{R}_E^T - \dot{z}\boldsymbol{A} + \boldsymbol{O}(\varepsilon_E),\end{aligned} \tag{95}$$

which follows from (25), (89), (88), (70) and $(A36)_2$. Note that the time derivative in this last form for the evolution of $\boldsymbol{A}$ is that appearing in the hypoelastoplastic form (A37) for $\boldsymbol{K}$. With the help of the identity

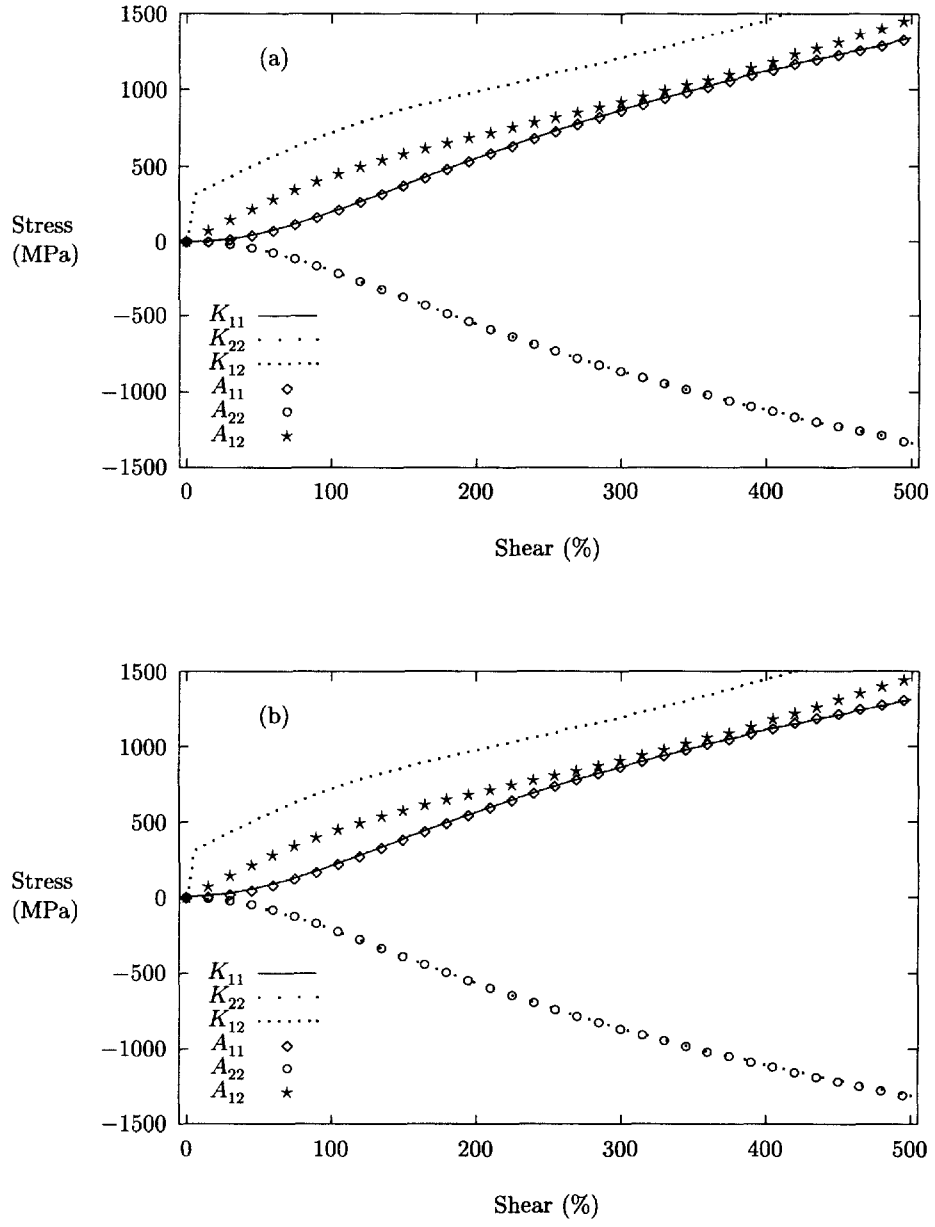


Fig. 2. Same as Fig. 1 except with $\boldsymbol{R}_{\mathrm{P}} = \boldsymbol{I}$ instead of $\boldsymbol{W}_{\mathrm{P}} = \boldsymbol{0}$ (part a), and the Green–Naghdi instead of Jaumann hypoelastoplastic model (part b).

$$\phi_{,\boldsymbol{K}} = \boldsymbol{E}\phi_{,\boldsymbol{M}}\boldsymbol{E}^{-1} \tag{96}$$

which follows from (21), we also have

$$\boldsymbol{E}\boldsymbol{D}_{\mathrm{P}}\boldsymbol{E}^{-1} = \lambda\phi_{,\boldsymbol{K}} \tag{97}$$

for the flow rule (34) relative to the current configuration and $\boldsymbol{K}$; in particular, this last result takes the form

$$\boldsymbol{R}_{\mathrm{E}}\boldsymbol{D}_{\mathrm{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} = \lambda\phi_{,\boldsymbol{K}} \tag{98}$$

in the case of elastic isotropy via (23). In the context of associated plasticity, (97) also arises from the alternative form

$$\delta_P = [\boldsymbol{K}-\boldsymbol{A}]\cdot\boldsymbol{E}\boldsymbol{D}_P\boldsymbol{E}^{-1} \tag{99}$$

for the plastic dissipation rate (density) with respect to the current configuration in the case of kinematic hardening alone from (21) and (22). So, together with the appropriate hypoelastic Jaumann- or Green–Naghdi-form [see (A37) and following discussion in the Appendix] for the evolution of $\boldsymbol{K}$, (95) yields the corresponding evolution of $\boldsymbol{A}$ in the case of linear, or more generally, Armstrong–Frederick, kinematic hardening.

Finally, we consider the case of simple shear for the algorithmic system $(77)_{1-3,5}$, with $(78)_{1,2}$ and (79), obtained in the current formulation. As discussed at the end of Section 6, the isotropy of ψ with respect to the intermediate configuration leads further to the special form $\boldsymbol{R}_P = \boldsymbol{I}$ for $\boldsymbol{R}_P$, and so the algorithmic relations (80), (81), in this case. The results corresponding to this case for the 11, 22 and 12 components of $\boldsymbol{K}$ and $\boldsymbol{A}$ from (82) are shown in Fig. 3(b). For comparison, the corresponding Oldroyd hypoelastoplastic case is plotted in Fig. 3(a), as based on $(A39)_3$ and the corresponding Oldroyd form

$$\dot{\boldsymbol{A}}-\boldsymbol{L}\boldsymbol{A}-\boldsymbol{A}\boldsymbol{L}^T = c\boldsymbol{R}_E\boldsymbol{D}_P\boldsymbol{R}_E^T-\dot{z}\boldsymbol{A} \tag{100}$$

for $\boldsymbol{A}$ when (98) is taken into account. The results for the shear stress and back stress in Fig. 3(b) are very close to those of the Oldroyd hypoelastoplastic case in Fig. 3(a), but not exactly the same. On the other hand, the normal stresses obtained in each case are different. That the corresponding back stress components in each of these latter two cases should both qualitatively and (almost) quantitatively be the same can be established on the basis of the relation

$$\begin{aligned}\dot{\boldsymbol{A}}-\boldsymbol{L}\boldsymbol{A}-\boldsymbol{A}\boldsymbol{L}^T &= \boldsymbol{R}_E[\dot{\boldsymbol{X}}-\boldsymbol{L}_P\boldsymbol{X}-\boldsymbol{X}\boldsymbol{L}_P^T]\boldsymbol{R}_E^T-2\boldsymbol{D}_E\boldsymbol{A}+\boldsymbol{O}(\varepsilon_E)\\ &= c\boldsymbol{R}_E\boldsymbol{D}_P\boldsymbol{R}_E^T-\dot{z}\boldsymbol{A}-2\boldsymbol{D}_E\boldsymbol{A}+\boldsymbol{O}(\varepsilon_E),\end{aligned} \tag{101}$$

which follows from (22), (25), (70) and (30). The small deviation of the back stress in the current formulation as compared to that of the Oldroyd hypoelastoplastic case can be attributed to the term $-2\boldsymbol{D}_E\boldsymbol{A}$, which is very small here. Given the almost exact quantitative agreement between the shear stress, as well as all back stress components, obtained from the two models, the difference between the corresponding normal stresses must be attributed to "small" differences in the elastic material behaviour modeled in each case. Indeed, as shown in Fig. 4(a), differences in the increase of K_{11} with increasing deformation between the two models in the purely elastic case [i.e., as based on (66) in the hyperelastic, and $(A19)_2$ in the hypoelastic, case] arise for (shear) deformations on the order of 0.5% or greater; on the other hand, as shown in Fig. 4(b), the corresponding elastic shear stresses from each model agree with each other exactly. As already mentioned, this is in contrast to the Jaumann- and Green–Naghdi-hypoelastic cases discussed above, in that these agree with (66) in all stress components up to about 100% shear.

10. DISCUSSION

The various thermodynamic formulations for non-linear kinematic hardening at large deformation discussed in this work can be compared with various existing mechanically-based models and formulations. In particular, the micromechanical approach of Dafalias (1983, 1985), Loret (1983), Aifantis (1987, 1995), Paulun and Pecherski (1987, 1992), Van der Giessen (1991), Tvergaard and Van der Giessen (1991), and others, as based on the plastic spin, for both linear and Armstrong–Frederick kinematic hardening, can also be formulated in the context of Chaboche's (1993) approach generalized to the intermediate configuration, as discussed in Section 8. Indeed, the correspondence follows in essence from the result (95) for the back stress, as well as compatibility between the form of the "plastic spin" used by these latter authors (with respect to the actual configuration) and the intermediate configuration quantity $\boldsymbol{W}_P$ appearing in the current work. To show this,

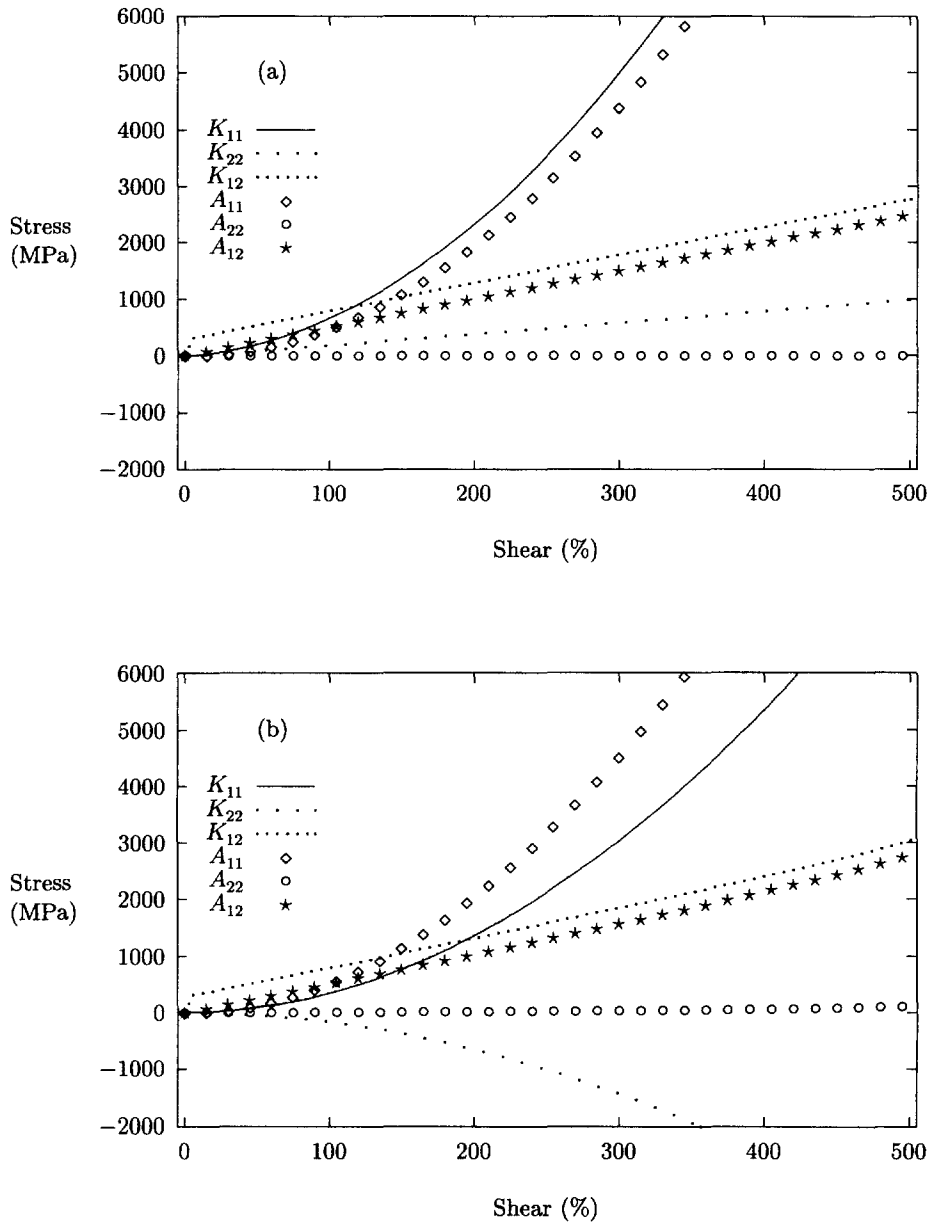


Fig. 3. Same as Fig. 1 except based on the system $(77)_{1\text{-}3,5}$, with $(78)_{1,2}$, (79) and (80), (81) (part b), compared with the Oldroyd hypoelastoplastic case (part a).

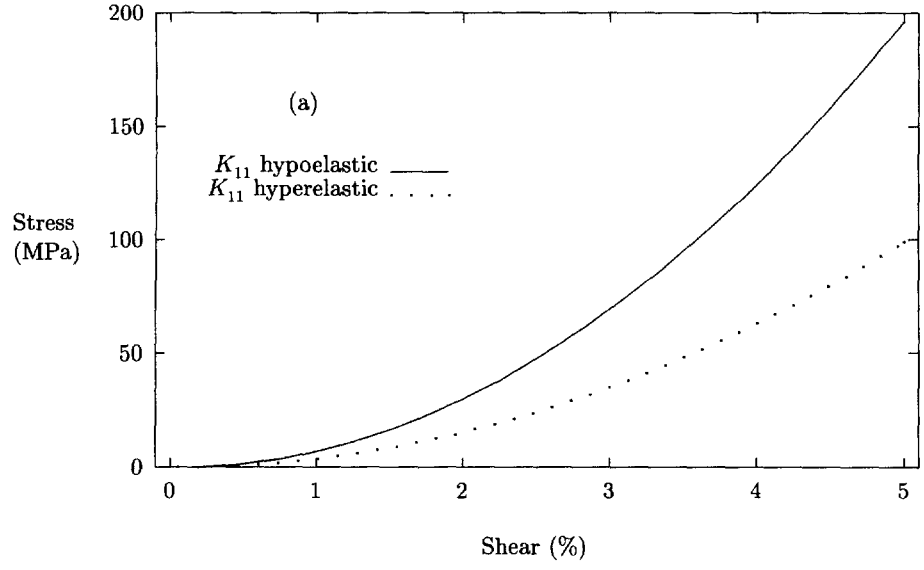
consider the multiplicative elastoplastic decomposition of $\boldsymbol{F}$ used by Dafalias (e.g., Dafalias, 1985, 1993), appropriate (only) for the case of isotropic elastic material behaviour, which takes the form

$$\boldsymbol{F} = \boldsymbol{V}_{\mathrm{D}}\boldsymbol{P}_{\mathrm{D}}, \tag{102}$$

where $\boldsymbol{V}_{\mathrm{D}}$ represents the (left) elastic (stretch) deformation, and $\boldsymbol{P}_{\mathrm{D}}$ the plastic part, of $\boldsymbol{F}$ in his formulation, and the subscript "D" stands for "Dafalias". Comparing (102) with the form (10) of this decomposition used here, we obtain the identifications

$$\boldsymbol{V}_{\mathrm{D}} \mathrel{\hat{=}} \boldsymbol{V}_{\mathrm{E}}, \boldsymbol{P}_{\mathrm{D}} \mathrel{\hat{=}} \boldsymbol{R}_{\mathrm{E}}\boldsymbol{P} \tag{103}$$

via (24). Time differentiation of $(103)_2$ then yields



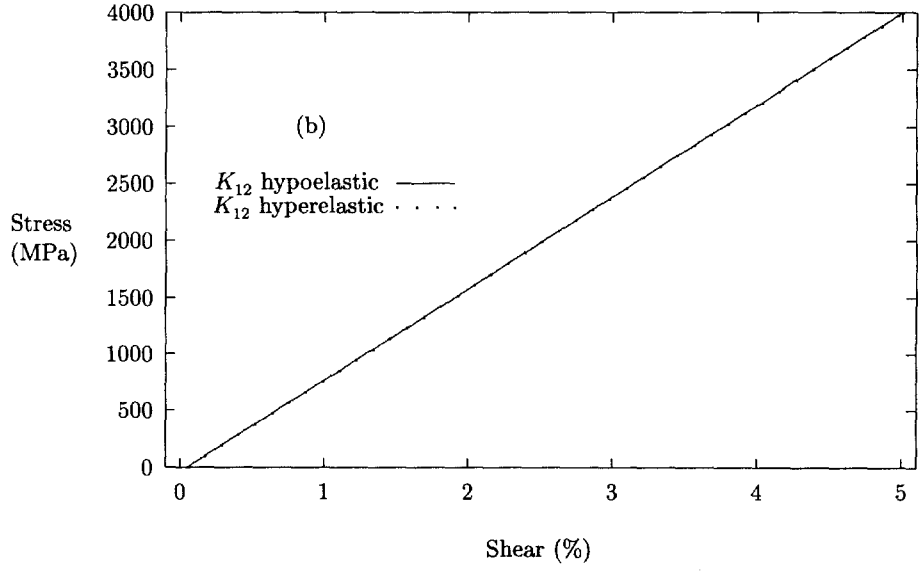


Fig. 4. Comparison of the K_{11} (part a) and K_{12} (part b) Kirchhoff stress components as based on the hyperelastic relation (66) and Oldroyd hypoelastic relation $(A19)_2$ up to 5% shear.

$$\dot{\boldsymbol{P}}_{\mathrm{D}} = \dot{\boldsymbol{R}}_{\mathrm{E}}\boldsymbol{P} + \boldsymbol{R}_{\mathrm{E}}\dot{\boldsymbol{P}} = \boldsymbol{\omega}\boldsymbol{P}_{\mathrm{D}} + \boldsymbol{R}_{\mathrm{E}}\dot{\boldsymbol{P}} \tag{104}$$

when we identify Dafalias's arbitrary spin $\boldsymbol{\omega}$ with $\dot{\boldsymbol{R}}_{\mathrm{E}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}$, i.e.,

$$\boldsymbol{\omega} \triangleq \dot{\boldsymbol{R}}_{\mathrm{E}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}. \tag{105}$$

With the help of his objective time derivative

$$\mathring{\boldsymbol{P}}_{\mathrm{D}} := \dot{\boldsymbol{P}}_{\mathrm{D}} - \boldsymbol{\omega}\boldsymbol{P}_{\mathrm{D}} \tag{106}$$

for $\boldsymbol{P}_{\mathrm{D}}$ with respect to $\boldsymbol{\omega}$, $(104)_2$ then yields

$$\mathring{\boldsymbol{P}}_{\mathrm{D}}\boldsymbol{P}_{\mathrm{D}}^{-1} = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{L}_{\mathrm{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} \tag{107}$$

via (14), and so the connection

$$\boldsymbol{W}_{\mathrm{PD}} := \mathrm{skw}(\mathring{\boldsymbol{P}}_{\mathrm{D}}\boldsymbol{P}_{\mathrm{D}}^{-1}) = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{W}_{\mathrm{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} \tag{108}$$

via (36) between Dafalias's form $\boldsymbol{W}_{\mathrm{PD}}$ of the plastic spin with respect to the current configuration and that $\boldsymbol{W}_{\mathrm{P}}$ used here with respect to the intermediate configuration. This last result shows that it is exactly Dafalias's form for the plastic spin which appears in the current configuration form (95) for the evolution of the back stress formulated thermodynamically in this work in the case that $\boldsymbol{P}$ is considered not to be a hardening variable.

As discussed briefly in the text, from the point of view of small incremental plastic deformations and small elastic strains, the results $(74)_{\mathrm{I}}$ and (76), central to the current work, show that $\boldsymbol{R}_{\mathrm{P}}$, $\boldsymbol{R}_{\mathrm{E}}$ and $\boldsymbol{W}_{\mathrm{P}}$, are, to $\boldsymbol{O}(\varepsilon_{\mathrm{E}})$, not all independent. In this context, then, constitutive assumptions involving $\boldsymbol{R}_{\mathrm{P}}$ and $\boldsymbol{W}_{\mathrm{P}}$ are not independent. In particular, as shown in the text, assuming $\boldsymbol{R}_{\mathrm{P}} = \mathbf{I}$ determines $\boldsymbol{W}_{\mathrm{P}}$ purely kinematically, which may suffice for some kinds of texture modeling, or modeling of the Swift effect, but not others (e.g., Bammann, 1990). Since $(74)_{\mathrm{I}}$ and (76) are independent of the assumption of elastic isotropy, note that they also hold in the more general case of elastic anisotropy, in which case the assumption $\boldsymbol{R}_{\mathrm{P}} = \boldsymbol{I}$ may make no sense (e.g., Svendsen, 1998b). In any case, this assumption represents the simplest means in the current hyperelastoplastic context of avoiding the famous shear stress oscillations during simple shear predicted by the Jaumann-based hypoelastoplasticity including kinematic hardening (e.g., Nagtegaal and DeJong, 1981).

Among hyperelastic-based models for large deformation elastoplasticity, Weber and Anand (1990) and Eterovic and Bathe (1990) assume $\boldsymbol{W}_{\mathrm{P}} = \mathbf{0}$ explicitly. As it turns out, Simo (1992) assumes in his principal axis-based formulation of hyperelastic elastoplasticity that the principle axes of the elastic left Cauchy–Green deformation are fixed during local integration. In the current context, this amounts to assuming that $\boldsymbol{R}_{\mathrm{E}} = \boldsymbol{R}_{\mathrm{Etr}}$, or equivalently from (76), that the plastic spin is identically zero, i.e., $\boldsymbol{W}_{\mathrm{P}} = \mathbf{0}$. As such, his formulation is, in the limit of small elastic strain, is also equivalent to Jaumann-based hypoelastoplasticity. As we have shown in this work, such an assumption is in fact unnecessary, as $\boldsymbol{W}_{\mathrm{P}}$ can be determined purely kinematically, i.e., at least in the case of metals when the assumption of small elastic strain applies.

The majority of existing numerical implementations of large-deformation hyperelastoplastic material models (e.g., Simon, 1988a, b, 1992; Simo and Ortiz, 1985; Weber and Anand, 1990; Simo and Miehe, 1992; Simo and Meschke, 1993; Miehe, 1996) presume isotropic elastic material behaviour, and are based in the current configuration, the major exception to this latter case being the work of Eterovic and Bathe (1990). With respect to the current configuration, the basic algorithmic form for the update of the elastic deformation is expressed in terms of the elastic left Cauchy–Green deformation tensor

$$\boldsymbol{B}_{\mathrm{E}} := \boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}. \tag{109}$$

The corresponding algorithmic form for $\boldsymbol{B}_{\mathrm{E}}$ can be obtained from (43) and the identity

$$\boldsymbol{F}\exp(\boldsymbol{A})\boldsymbol{F}^{-1} = \exp(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{-1}), \tag{110}$$

i.e.,

$$\boldsymbol{B}_{\mathrm{Etr}} = \boldsymbol{E}\exp(2\tau\boldsymbol{D}_{\mathrm{P}})\boldsymbol{E}^{\mathrm{T}} = \exp(2\tau\boldsymbol{E}\boldsymbol{D}_{\mathrm{P}}\boldsymbol{E}^{-1})\boldsymbol{B}_{\mathrm{E}} = \exp(2\gamma\phi_{,\boldsymbol{K}})\boldsymbol{B}_{\mathrm{E}} \tag{111}$$

via (52) and (97). By working with the general algorithmic form (43) from the start, there is no need to introduce an evolution for $\boldsymbol{B}_{\mathrm{E}}$ and so a "split" of this relation into kinematic and constitutive parts, as done by Simo (1992). Although it is certainly possible to implement models for isotropic hyperelastoplasticity with kinematic hardening on the basis

of (111) (i.e. as shown by the results (95) and (101) of the previous section), the numerical integration may be easier to carry out (depending on the model) with respect to the intermediate configuration. Furthermore, a formulation and implementation with respect to this latter configuration facilitates generalizations of the model to, e.g., anisotropic elastic material behaviour, the subject of future work.

Acknowledgements—We thank Stefanie Reese and Charalamps Tsakmakis for helpful comments and discussions on many aspects of this work.

REFERENCES

Aifantis, E. (1987) On the physics of plastic deformation. *International Journal of Plasticity* **3,** 211–247.
Aifantis, E. (1995) From micro- to macro-plasticity: the scale-invariance approach. *J. Eng. Mat. Tech.* **117,** 352–355.
Armstrong, P. J. and Frederick, C. O. (1966) A mathematical representation of the multiaxial Bauschinger effect. General Electricity Generating Board Report No. RD/B/N/731. o. O.
Arndt, S., Klingbeil, D. and Svendsen, B. (1997) On the simulation of warm pre-stressing and ductile crack extension by constitutive modelling. *Transactions of the 14th International Conference on Structural Mechanics in Reactor Technology*, **4,** pp. 689–696.
Atluri, S. N. (1984) On constitutive relations at finite strain: hypoelasticity and elastoplasticity with isotropic or kinematic hardening. *Comp. Methods Appl. Mech. Eng.* **43,** 137–171.
Bammann, D. J. (1990) Modeling temperature and strain rate dependent large deformations of metals. *Applied Mechanics Review* **43,** S312–S319.
Bammann, D. J. and Johnson, G. C. (1987) On the kinematics of finite-deformation plasticity. *Acta Mechanica* **70,** 1–13.
Bertram, A. (1993) Description of finite inelastic deformations. In *MECAMAT '92, "Multiaxial Plasticity"*, ed. A. Benallal, R. Billardon and D. Marquis, pp. 821–835.
Chaboche, J.-L. (1993) Cyclic viscoplastic constitutive relations, Part I: a thermodynamically-consistent formulation. *Journal of Applied Mechanics* **60,** 813–828.
Dafalias, Y. F. (1983) A missing link in the macroscopic constitutive formulation of large plastic deformations, Chap. 8. In *Plasticity Today*, ed. A. Sawczuk and G. Bianchi, pp. 135–151. Elsevier Applied Science, U.K.
Dafalias, Y. F. (1985) The plastic spin. *Journal of Applied Mechanics* **52,** 865–871.
Dafalias, Y. F. (1993) On multiple spins and texture development. Case study: kinematic and orthotropic hardening. *Acta Mechanica* **100,** 171–194.
Dienes, J. K. (1979) On the analysis of rotation and stress rate in deforming bodies. *Acta Mechanica* **32,** 217–232.
Dienes, J. K. (1986) A discussion of material rotation and stress rate. *Acta Mechanica* **65,** 1–11.
Dogui, A. and Sidoroff, F. (1985) Kinematic hardening in large elastoplastic strain. *Eng. Frac. Mech.* **21,** 685–695.
Eterovic, A. L. and Bathe, K. J. (1990) A hyperelastic-based large strain elastoplastic constitutive formulation with combined isotropic-kinematic hardening using logarithmic stresses and strain measures. *International Journal of Numerical Methods in Engineering* **30,** 1099–1115.
Green, A. E. and McInnis, B. C. (1967) Generalized hypoplasticity. *Proceedings of the Royal Society of Edinburgh* **A57,** 220–230.
Green, A. E. and Naghdi, P. M. (1965) A general theory of the elastic–plastic continuum. *Arch. Rat. Mech. Anal.* **A18,** 251–281.
Haupt, P. (1995) On the thermodynamic representation of viscoplastic material behaviour. *Proceedings of the AMSE Materials Division* MD **69-1,** 503–519.
Haupt, P. and Tsakmakis, Ch. (1986) On kinematic hardening and large plastic deformations. *International Journal of Plasticity* **2,** 279–293.
Haupt, P. and Tsakmakis, Ch. (1989) On the application of dual variables in continuum mechanics. *Continuum Mech. Thermodyn.* **1,** 165–196.
Loret, B. (1983) On the effects of plastic rotation in the finite deformation of anisotropic elastoplastic materials. *Mech. Materials* **2,** 287–304.
Lubliner, J. (1973) On the structure of rate equations in materials with internal variables. *Acta Mechanica* **17,** 109–119.
Lubliner, J. (1984) A maximum-dissipation principle in generalized plasticity. *Acta Mechanica* **52,** 225–250.
Lubliner, J. (1986) On normality rules in large-deformation plasticity. *Mech. Mat.* **5,** 29–34.
Mandel, J. (1972) Plasticité classique et viscoplasticité. *CISM Course*. Springer-Verlag.
Mandel, J. (1974) Thermodynamics and plasticity. In *Foundations of Continuum Thermodynamics*, ed. J. J. Delgado Domingos, M. N. R. Nina and J. H. Whitelaw. MacMillan.
Maugin, G. (1992) *The Thermodynamics of Plasticity and Fracture*. Cambridge University Press.
Miehe, C. (1994) On the representation of Prandtl–Reuss tensors within the framework of multiplicative elastoplasticity. *International Journal of Plasticity* **10,** 609–621.
Miehe, C. (1996) Exponential map algorithm for stress updates in anisotropic multiplicative elastoplasticity for single crystals. *International Journl of Numerical Methods in Engineering* **39,** 3367–3390.
Nagtegaal, J. C. and DeJong, J. E. (1981) Some aspects of nonisotropic work hardening in finite strain plasticity. In *Proceedings of the Workshop on Plasticity of Metals at Finite Strain*, pp. 65–106. Div. Appl. Mech. Standford University.
Noll, W. (1967) Material uniform inhomogeneous material bodies. *Arch. Rat. Mech. Anal.* **27,** 1–32.
Noll, W. (1972) A new mathematical theory of simple materials. *Arch. Rat. Mech. Anal.* **48,** 1–50.

Paulun, J. E. and Pecherski, R. B. (1987) On the application of the plastic spin concept for the description of anisotropic hardening in finite deformation plasticity. *International Journal of Plasticity* **3,** 303–314.
Paulun, J. E. and Pecherski, R. B. (1992) On the relation for the plastic spin. *Arch. Appl. Mech.* **62,** 376–385.
Perić, D. (1992) On consistent stress rates in solid mechanics: computational implications. *International Journal of Numerical Methods in Engineering* **33,** 799–817.
Sievert, R. (1997) Considerations on the back-stress at large strain and non-isothermal loading. Report BAM-V.31 97/2. Federal Institute for Materials Research and Testing, Berlin.
Simo, J. C. (1988a) A framework for finite strain elastoplasticity based on the multiplicative decomposition and hyperelastic relations. Part I: formulation. *Comp. Methods Appl. Mech. Eng.* **66**(2), 199–219.
Simo, J. C. (1988b) A framework of finite strain elastoplasticity based on the multiplicative decomposition and hyperelastic relations. Part II: computational aspects. *Comp. Methods Appl. Mech. Eng.* **66**(3), 1–31.
Simo, J. C. (1992) Algorithms for static and dynamic multiplicative plasticity that preserve the classical return mappings schemes of the infinitesimal theory. *Comp. Methods Appl. Mech. Eng.* **99,** 61–112.
Simo, J. C. and Meschke, G. (1993) A new class of algorithms for classical plasticity extended to finite strains. Application to geomaterials. *Comp. Mech.* **11,** 253–278.
Simo, J. C. and Miehe, C. (1991) Coupled associative thermoplasticity at finite strains. Formulation, numerical analysis and implementation. *Comp. Methods Appl. Mech. Eng.* **98,** 41–104.
Simo, J. C. and Ortiz, M. (1985) A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive relations. *Comp. Methods Appl. Mech. Eng.* **49,** 221–245.
Simo, J. C. and Pister, M. (1984) Remarks on rate constitutive equations for finite deformation problems: computational implications. *Comp. Methods Appl. Mech. Eng.* **46,** 201–215.
Svendsen, B. (1998a) A thermodynamic formulation of finite-deformation elastoplasticity with hardening based on the concept of material isomorphism. *International Journal of Plasticity*, in press.
Svendsen, B. (1998b) On material isomorphism, material symmetry and material uniformity in finite-deformation elastoplasticity. *International Journal of Engineering Science*, submitted.
Svendsen, B. and Tsakmakis, Ch. (1994) On the generalization of constitutive relations from small to large deformations. *Z. angew. Math. Mech.* **74,** T320–T322.
Szabó, L. and Balla, M. (1992) Comparison of some stress rates. *International Journal of Solids and Structures* **25,** 279–297.
Truesdell, C. and Noll, W. (1992) *The Non-Linear Field Theories of Mechanics*, 2nd edn. Springer.
Tvergaard, V. and Van der Giessen, E. (1991) Effect of plastic spin on localization predictions for a porous ductile material. *Journal of the Mechanics and Physics of Solids* **39,** 763–781.
Tsakmakis, Ch. (1996) Kinematic hardening rules at large deformations, Part I: a constitutive approach. *Cont. Mech. Thermodyn.* **8,** 215–231.
Van der Giessen, E. (1991) Micromechanical and thermodynamic aspects of the plastic spin. *International Journal of Plasticity* **7,** 365–386.
Weber, G. and Anand, L. (1990) Finite deformation constitutive equations and a time integration procedure for isotropic, hyperelastic–viscoelastic solid. *Comp. Methods Appl. Mech. Eng.* **79,** 173–202.
Zavaliangos, A. and Anand, L. (1993) Thermo-elasto-viscoplasticity of isotropic porous metal. *Journal of the Mechanics and Physics of Solids* **41,** 1087–1118.

APPENDIX: CLASSICAL HYPOELASTIC MODELS

For completeness, we briefly derive in this appendix the classical hypoelastic cases relevant to elastoplasticity at large deformation in metals discussed in the text. For an alternative, more detailed formulation on this, we refer the reader to Perić (1992). According to Truesdell and Noll (1965), hypoelasticity can represent the material response of an elastic material only when (1), the material possesses a natural, i.e., stress-free, reference state, and (2), the response of the material to small strain "perturbations" away from this state is isotropic. As such, any "derivation" of such relations from finite elastic relations relies in general on the assumptions of (1), isotropic material behaviour, and (2), the existence of a natural state, as well as on (3), a physical linearization of the finite relation via time-differentiation, and (4), reduction of result to the case of small "deformation".

To begin, it is instructive to first look at the purely elastic case, i.e., when $\boldsymbol{P} = \boldsymbol{I}$. We begin by introducing the polar decomposition

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{U} \tag{A1}$$

of $\boldsymbol{F}$. The time derivative of this last relation yields as usual

$$\boldsymbol{L}_{\boldsymbol{F}} = \boldsymbol{W}_{\boldsymbol{R}} + \boldsymbol{R}\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{R}^{\mathrm{T}}, \tag{A2}$$

where we have introduced the notation

$$\begin{aligned} \boldsymbol{L}_{\boldsymbol{A}} &:= \dot{\boldsymbol{A}}\boldsymbol{A}^{-1}, \\ \boldsymbol{D}_{\boldsymbol{A}} &:= \mathrm{sym}(\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}), \\ \boldsymbol{W}_{\boldsymbol{A}} &:= \mathrm{skw}(\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}), \end{aligned} \tag{A3}$$

for any time-dependent second-order tensor $\boldsymbol{A}$; in particular, we have then $\boldsymbol{L}_{\boldsymbol{F}} = \boldsymbol{L}$, $\boldsymbol{D}_{\boldsymbol{F}} = \boldsymbol{D}$ and $\boldsymbol{W}_{\boldsymbol{F}} = \boldsymbol{W}$ with

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \boldsymbol{D} + \boldsymbol{W}, \tag{A4}$$

as well as $\boldsymbol{L}_{\boldsymbol{R}} = \boldsymbol{W}_{\boldsymbol{R}}$. Consider next the Kirchhoff stress in the basic material frame-indifferent forms

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{R}\mathscr{K}(\boldsymbol{U})\boldsymbol{R}^{\mathrm{T}} \\ &= \boldsymbol{F}\mathscr{S}(\boldsymbol{C})\boldsymbol{F}^{\mathrm{T}} \end{aligned} \tag{A5}$$

relative to $\boldsymbol{U}$ and $\boldsymbol{C}$, respectively with

$$\mathscr{K}(\boldsymbol{U}) = \boldsymbol{U}\mathscr{S}(\boldsymbol{C})\boldsymbol{U}. \tag{A6}$$

As is the case in hypoelasticity, we assume that the reference configuration with respect to which (A5) are expressed is stress-free, i.e.,

$$\mathscr{K}(\boldsymbol{I}) = \boldsymbol{0} = \mathscr{S}(\boldsymbol{I}). \tag{A7}$$

Taking now time derivatives of (A5) yields their physically-linear forms

$$\begin{aligned} \delta_{\boldsymbol{R}}\boldsymbol{K} &= \boldsymbol{R}(D\mathscr{K})(\boldsymbol{U})[\boldsymbol{R}^{\mathrm{T}}(\delta_{\boldsymbol{R}}\boldsymbol{V})\boldsymbol{R}]\boldsymbol{R}^{\mathrm{T}} \\ \delta_{\boldsymbol{F}}\boldsymbol{K} &= 2\boldsymbol{F}(D\mathscr{S})(\boldsymbol{C})[\boldsymbol{F}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{F}}\boldsymbol{F}]\boldsymbol{F}^{\mathrm{T}}. \end{aligned} \tag{A8}$$

In these last relations, the Fréchet derivatives $(D\mathscr{K})(\boldsymbol{U})$ and $(D\mathscr{S})(\boldsymbol{C})$ of $\mathscr{K}$ and $\mathscr{S}$ represent the corresponding elasticity tensors. In addition, we have introduced the notation

$$\delta_{\boldsymbol{A}}\boldsymbol{K} := \boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{K}\boldsymbol{A}^{-\mathrm{T}})^{\cdot}\boldsymbol{A}^{\mathrm{T}} \tag{A9}$$

for the objective time derivative of $\boldsymbol{K}$ with respect to any time-dependent invertible tensor $\boldsymbol{A}$. In particular,

$$\begin{aligned} \delta_{\boldsymbol{R}}\boldsymbol{V} &= \boldsymbol{R}\dot{\boldsymbol{V}}\boldsymbol{R}^{\mathrm{T}} \\ &= \dot{\boldsymbol{V}} - [\boldsymbol{W}_{\boldsymbol{R}}, \boldsymbol{V}] \\ &= \boldsymbol{D}_{\boldsymbol{F}}\boldsymbol{V} + (\boldsymbol{W}_{\boldsymbol{F}} - \boldsymbol{W}_{\boldsymbol{R}})\boldsymbol{V} \end{aligned} \tag{A10}$$

represents the Green–Naghdi derivative of $\boldsymbol{V}$, which follows from (A1) and (A2). Further, the identity

$$\dot{\boldsymbol{C}} = 2\boldsymbol{F}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{F}}\boldsymbol{F} \tag{A11}$$

has also been used to obtain $(A8)_2$.

For simplicity, we restrict attention here to the case of small total strain as defined by

$$\begin{aligned} \boldsymbol{U} &= \boldsymbol{I} + \boldsymbol{O}(\varepsilon) \\ \boldsymbol{F} &= \boldsymbol{R} + \boldsymbol{O}(\varepsilon) \end{aligned} \tag{A12}$$

via (A1), with

$$\varepsilon := |\ln \boldsymbol{V}| = |\ln \boldsymbol{U}|. \tag{A13}$$

This is in contrast to the case considered in the text, in which only the deviatoric part (i.e., of $\ln \boldsymbol{U}_{\mathrm{E}}$) was considered small. Going a step further than (A12) would be of course to assume geometric linearity, in which both strains and rotations are small. Indeed, in this latter case, the displacement gradient $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$ is assumed small, something which holds only when both $\boldsymbol{U}$ and $\boldsymbol{R}$ are small. In particular, (A12) yields

$$\boldsymbol{L}_{\boldsymbol{U}} = \dot{\boldsymbol{U}} + \boldsymbol{O}(\varepsilon) \tag{A14}$$

via (A3). This last approximation, as well as (A2) and the symmetry of $\dot{\boldsymbol{U}}$, imply

$$\boldsymbol{W}_{\boldsymbol{U}} = \boldsymbol{0} + \boldsymbol{O}(\varepsilon), \tag{A15}$$

and so

$$\boldsymbol{W}_{\boldsymbol{F}} = \boldsymbol{W}_{\boldsymbol{R}} + \boldsymbol{O}(\varepsilon) \tag{A16}$$

from (A2). In turn, this last result leads to

$$\delta_{\boldsymbol{R}}\boldsymbol{K} = \dot{\boldsymbol{K}} - [\boldsymbol{W}_{\boldsymbol{F}}, \boldsymbol{K}] + \boldsymbol{O}(\varepsilon). \tag{A17}$$

Consequently, we may replace the Green–Naghdi derivative $\delta_{\boldsymbol{R}}\boldsymbol{K}$ with its Jaumann counterpart to $\boldsymbol{O}(\varepsilon)$ in $(A8)_1$. Using these results in (A8) and (A10) yields

$$\begin{aligned} \delta_{\boldsymbol{R}}\boldsymbol{K} &= \boldsymbol{R}(D\mathscr{K})(\boldsymbol{I})[\boldsymbol{R}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{F}}\boldsymbol{R}]\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon) \\ \delta_{\boldsymbol{F}}\boldsymbol{K} &= 2\boldsymbol{R}(D\mathscr{S})(\boldsymbol{I})[\boldsymbol{R}^{\mathrm{T}}\boldsymbol{D}_{\boldsymbol{F}}\boldsymbol{R}]\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon). \end{aligned} \tag{A18}$$

Finally, in the case of isotropic material behaviour, these reduce to the Green–Naghdi and Oldroyd hypoelastic forms

$$\delta_{\boldsymbol{R}}\boldsymbol{K} = \boldsymbol{H}_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{F}}],$$
$$\delta_{\boldsymbol{F}}\boldsymbol{K} = \boldsymbol{H}_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{F}}], \tag{A19}$$

respectively, valid to $\boldsymbol{O}(\varepsilon)$, with

$$\boldsymbol{H}_{\mathrm{E}} := (D\mathscr{K})(\boldsymbol{I}) = 2(D\mathscr{S})(\boldsymbol{I}), \tag{A20}$$

the second equality following from (A6) and the condition (A7). Here,

$$\boldsymbol{H}_{\mathrm{E}} = \lambda(\boldsymbol{I}\otimes\boldsymbol{I}) + 2\mu\boldsymbol{I} \tag{A21}$$

represents the usual constant, isotropic, fourth-order elasticity tensor appearing from the start in the case of hypoelasticity; in the current context, it arises with respect to the natural state $\boldsymbol{U} = \boldsymbol{I}$ in the context of (A12).

These last results can be applied in particular to the case of isotropic hyperelasticity. Indeed, in this case, we have

$$\mathscr{K} = \psi_{,\ln \boldsymbol{U}} = D\psi, \tag{A22}$$

with

$$\psi = \psi(\ln \boldsymbol{U}) = \psi(I_1, I_2, I_3) \tag{A23}$$

for ψ analogous to (59) with (60). The Fréchet derivative of (A22) yields then the connection

$$(D\mathscr{K})(\boldsymbol{U}) = (D^2\psi)(\ln \boldsymbol{U})(D\ln)(\boldsymbol{U}) \tag{A24}$$

between the corresponding elasticity tensors; in particular,

$$\boldsymbol{H}_{\mathrm{E}} = (D\mathscr{K})(\boldsymbol{I}) = (D^2\psi)(\boldsymbol{0})(D\ln)(\boldsymbol{I}) = \psi_{,\ln\boldsymbol{U}\ln\boldsymbol{U}}|_{\ln\boldsymbol{U}=\boldsymbol{0}} = \psi_{,I_1I_1}|_{\ln\boldsymbol{U}=\boldsymbol{0}}\boldsymbol{I}\otimes\boldsymbol{I} + 2\psi_{,I_2}|_{\ln\boldsymbol{U}=\boldsymbol{0}}\boldsymbol{I} \tag{A25}$$

follows from (A12), (A22) and (A23), with $(D\ln)(\boldsymbol{I}) = \boldsymbol{I}$. Of course, $\boldsymbol{C}_{\mathrm{E}}$ as defined in (67) in the text reduces to $\boldsymbol{H}_{\mathrm{E}}$ when we assume $\mathrm{sph}(\ln \boldsymbol{U})$ is small, as we are doing in this appendix.

With these basic results, we now turn to hypoelastoplasticity. By analogy with (A5) and (A19), time differentiation of the elastoplastic forms

$$\boldsymbol{K} = \boldsymbol{R}_{\mathrm{E}}\mathscr{K}(\boldsymbol{U}_{\mathrm{E}})\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}$$
$$= \boldsymbol{E}\mathscr{S}(\boldsymbol{C}_{\mathrm{E}})\boldsymbol{E}^{\mathrm{T}} \tag{A26}$$

for the Kirchhoff stress, together with the assumption of small elastic strain, yields the incremental forms

$$\delta_{\boldsymbol{R}_{\mathrm{E}}}\boldsymbol{K} = \boldsymbol{H}_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{E}}],$$
$$\delta_{\boldsymbol{F}_{\mathrm{E}}}\boldsymbol{K} = \boldsymbol{H}_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{E}}], \tag{A27}$$

in this context, again to $\boldsymbol{O}(\varepsilon_{\mathrm{E}})$, where for simplicity we now consider $\varepsilon_{\mathrm{E}} = |\ln \boldsymbol{V}_{\mathrm{E}}| = |\ln \boldsymbol{U}_{\mathrm{E}}|$ to be "small", and here

$$\boldsymbol{H}_{E} := (D\mathscr{K})(\boldsymbol{I}) = 2(D\mathscr{S})(\boldsymbol{I}) = \varphi_{\mathrm{E},\ln\boldsymbol{U}_{\mathrm{E}}\ln\boldsymbol{U}_{\mathrm{E}}}|_{\ln\boldsymbol{U}_{\mathrm{E}}=\boldsymbol{0}} \tag{A28}$$

from (A20) and (A25). Similarly, the analogy with (A12) implies

$$\boldsymbol{U}_{\mathrm{E}} = \boldsymbol{I} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}),$$
$$\boldsymbol{E} = \boldsymbol{R}_{\mathrm{E}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \tag{A29}$$

via (24). Likewise,

$$\boldsymbol{L}_{\boldsymbol{U}_{\mathrm{E}}} = \dot{\boldsymbol{U}}_{\mathrm{E}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) \tag{A30}$$

follows by analogy with (A14), and so the relations

$$\boldsymbol{W}_{\mathrm{E}} = \boldsymbol{W}_{\boldsymbol{R}_{\mathrm{E}}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) \tag{A31}$$

and

$$\boldsymbol{W}_{\boldsymbol{U}_{\mathrm{E}}} = \boldsymbol{0} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \tag{A32}$$

respectively, in the case of small elastic strain.

The basic connection between the total, elastic and plastic deformation rates and spins is obtained via time differentiation of the elastic deformation tensor (10) and rearrangement, i.e.,

$$\boldsymbol{L}_{\boldsymbol{E}} = \boldsymbol{L}_{\boldsymbol{F}} - \boldsymbol{E}\boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{E}^{-1}. \tag{A33}$$

Reduction of this last relation to hypoelastoplastic form is contingent on the result

$$\boldsymbol{E}\boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{E}^{-1} = \boldsymbol{R}_{\mathrm{E}}\boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) \tag{A34}$$

via (29). Substitution of (A34) into (A33) then yields

$$\boldsymbol{L}_{\boldsymbol{E}} = \boldsymbol{L}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \tag{A35}$$

from which we obtain

$$\begin{aligned} \boldsymbol{D}_{\boldsymbol{E}} &= \boldsymbol{D}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{D}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \\ \boldsymbol{W}_{\boldsymbol{E}} &= \boldsymbol{W}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{W}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}). \end{aligned} \tag{A36}$$

Using the results (A31) and (A36) in $(A27)_1$ yields the alternative form

$$\dot{\boldsymbol{K}} - [\boldsymbol{W}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{W}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}, \boldsymbol{K}] = H_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{D}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}] + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) \tag{A37}$$

of $(A27)_1$. This last result contains in fact the usual Jaumann and Green–Naghdi forms for hypoelastoplasticity. Indeed, in view of the associated form (98) for $\boldsymbol{R}_{\mathrm{E}}\boldsymbol{D}_{\mathrm{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}$, if we assume $\boldsymbol{W}_{\boldsymbol{P}} = \boldsymbol{0}$, the Jaumann form is obtained. As discussed in the text, it has long been known that neglecting the "plastic spin" $\boldsymbol{W}_{\mathrm{P}}$ leads to Jaumann-like behaviour in combination with kinematic hardening (e.g., Dafalias, 1985); as well, this is consistent with the hyperelastic-based results for simple shear discussed in the text. Alternatively, if we assume $\boldsymbol{R}_{\mathrm{P}} = \boldsymbol{I}$, the results (74) (assuming again that $|\mathrm{sph}(\ln \boldsymbol{U}_{\mathrm{E}})| \ll 1$) lead to

$$\boldsymbol{W}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{W}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} = \boldsymbol{W}_{\boldsymbol{F}} - \boldsymbol{R}\boldsymbol{W}_{\boldsymbol{U}}\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) = \boldsymbol{W}_{\boldsymbol{R}} + \boldsymbol{O}(\varepsilon_{rE}), \tag{A38}$$

in which case (A37) reduces to the Green–Naghdi form for hypoelastoplasticity. This result is also completely consistent with our hyperelastic-based approach discussed in the text. Finally, the usual Oldroyd form

$$\begin{aligned} \delta_{\boldsymbol{F}}\boldsymbol{K} &= \boldsymbol{R}_{\mathrm{E}}\boldsymbol{L}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{R}_{\mathrm{E}}\boldsymbol{L}_{\boldsymbol{P}}^{\mathrm{T}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}} + \delta_{\boldsymbol{F}_{\mathrm{E}}}\boldsymbol{K} \\ &= \delta_{\boldsymbol{F}_{\mathrm{E}}}\boldsymbol{K} + \boldsymbol{O}(\varepsilon_{\mathrm{E}}) \\ &= H_{\mathrm{E}}[\boldsymbol{D}_{\boldsymbol{F}} - \boldsymbol{R}_{\mathrm{E}}\boldsymbol{D}_{\boldsymbol{P}}\boldsymbol{R}_{\mathrm{E}}^{\mathrm{T}}] + \boldsymbol{O}(\varepsilon_{\mathrm{E}}), \end{aligned} \tag{A39}$$

for hypoelastoplasticity arises from (A35), $(A27)_2$, $(A36)_2$, and the fact that $\boldsymbol{K} = \boldsymbol{O}(\varepsilon_{\mathrm{E}})$.